M7210 Lecture 13 Friday September 21, 2012

A. Orthogonality and orthonormal bases

Let V be an inner-product space (real or Hermitian). The inner product is denoted $\langle u, v\rangle$ and the associated norm is $||v|| := \sqrt{\langle v, v\rangle}$.

Definition. We say $u, v \in V$ are *orthogonal*—and we write $u \perp v$—if $\langle u, v\rangle = 0$. We say u is a *unit vector* if $||u|| = 1$. An *orthonormal set* is a set of unit vectors that are pairwise orthogonal.

Exercise. Suppose $w_1, \ldots, w_k$ are pairwise orthogonal. Then

$$||w_1 + \cdots + w_k||^2 = ||w_1||^2 + \cdots + ||w_k||^2.$$

Hint: *Use the* relevant definitions*, Luke.*

Lemma. *Any set of non-zero vectors that are pairwise orthogonal is independent.*

Proof. Suppose $w_1, \ldots, w_k \in V \setminus \{0\}$ are pairwise orthogonal. If $c_1w_1 + \cdots + c_kw_k = 0$, then $||c_1w_1 + \cdots + c_kw_k||^2 = |c_1|^2||w_1||^2 + \cdots + |c_k|^2||w_k||^2 = 0$, so $|c_i|\,||w_i|| = 0$ for each i, and since $||w_i|| \neq 0$, each $c_i = 0$. /////

Observe that if u is a unit vector and v is any vector, then $u \perp \Big(v - \langle v, u\rangle\, u\Big)$. The demonstration is instructive:

$$\Big\langle u, v - \langle v, u\rangle\, u \Big\rangle = \langle u, v\rangle - \Big\langle u, \langle v, u\rangle\, u \Big\rangle = \langle u, v\rangle - \overline{\langle v, u\rangle}\langle u, u\rangle = 0.$$

Many people call $\langle v, u\rangle\, u$ the *projection of* v *onto* u and call $v - \langle v, u\rangle\, u$ the *component of* v *orthogonal to* u.

Lemma. *Suppose* $\{u_1, \ldots, u_k\} \subseteq V$ *is orthonormal and* $v \in V$*. Let* $w = \sum_{i=1}^{k}\langle v, u_i\rangle u_i$*. Then*

$$u_j \perp (v - w)\,, \quad \text{for each } j = 1, \ldots, k, \tag{1}$$

and

$$w \perp (v - w). \tag{2}$$

Proof. First, observe that when we expand $\langle u_j, w\rangle$ using conjugate-linearity in the second variable, every term but the j^{th} vanishes, the reason being that $\langle u_i, u_j\rangle = 0$ unless $i = j$. Thus,

$$\langle u_j, w\rangle = \Big\langle u_j, \langle v, u_j\rangle u_j \Big\rangle = \langle u_j, v\rangle,$$

and equation (1) follows. Using this and linearity in the first variable, we get:

$$\langle w, w\rangle = \langle w, v\rangle.$$

Equation (2) follows. /////

The lemma shows that any $v \in V$ can be written as the sum of an element w of a the subspace U spanned by $u_1, \dots, u_k$ plus an element $v - w$ that is orthogonal to every element of U. The manner in which this can be done is **unique**, for if $v = c_1 u_1 + \cdots + c_k u_k + z$, where $e_i \perp z$ for each $i = 1, \dots, k$, then $\langle v, u_i \rangle = c_i$, for $i = 1, \dots, k$, and consequently $z = v - w$.

Gram-Schmidt orthogonalization. Suppose $U_k \subseteq V$ is spanned by an orthonormal set $\{u_1, \dots, u_k\} \subseteq V$, and U_k is not all of V. Let v be any element of $V \setminus U_k$, and select w as in the lemma. Let $c = ||v - w||$ and let $u_{k+1} := c^{-1}(v - w)$. Then $\{u_1, \dots, u_k, u_{k+1}\}$ is an orthonormal set. Let U_{k+1} be the subspace it spans. Obviously, we can continue in this manner as long as $U_m \neq V$.

Orthogonal Projection. Given any subspace U of V, the ideas in the lemma enable us decompose V into sum of orthogonal subspaces

$$V = U \oplus U^{\perp}.$$

Select an orthogonal basis $\{u_1, \dots, u_k\}$ for U, and then extend it to an orthogonal basis $\{u_1, \dots, u_k, u_{k+1}, \dots, u_n\}$ for V, and let $U^{\perp}$ be the subspace spanned by $\{u_{k+1}, \dots, u_n\}$.

Exercise. Write the projection and injection maps for this decomposition in terms of the selected basis.

B. Inner products and the dual space

The material in this subsection is not dependent on the results above concerning orthogonality. Recall that V' denotes the dual space of V:

$$V' := \{ \ell : V \to \mathbb{F} \mid \ell \text{ linear} \}.$$

The real case

Proposition. *Suppose V is a real inner-product space. The function*

$$\phi : V \to V'; v \mapsto \phi_v$$

defined by

$$\phi_v(u) := \langle u, v \rangle$$

(i.e., $\phi_v = \langle \cdot, v \rangle$) is a bijective linear map.

Proof. By linearity in the second variable, $\phi_{cv} = c\phi_v$ and $\phi_{v+w} = \phi_v + \phi_w$. Thus ϕ is a linear map. Since $\phi_v(v) \neq 0$ when $v \neq 0$, the kernel of ϕ is 0. Since $\dim V = \dim V'$, ϕ is bijective. /////

The Hermitian case. For Hermitian inner-product spaces, the map described in the proposition above is not linear (and—if you thought you might rescue yourself by switching variables—the map $v \mapsto \langle v, \cdot \rangle$ does not have values in V'). One way to deal with this is to introduce the so-called *anti-module of* V, denoted $\widetilde{V}$, which is a vector space with the same elements and the same addition as V, but in which scalar multiplication differs by conjugation. In order to describe the scalar multiplication in $\widetilde{V}$, it is convenient to adopt a notation that makes it evident which space we are in. When we want v to be understood as an element of $\widetilde{V}$, we draw a tilde over it. Now, we can be explicit about the scalar multiplicationin $\widetilde{V}$. It is defined by $c\tilde{v} = \widetilde{\overline{c}v}$.

Exercise. Show that a subset $\{v_1, \ldots, v_k\}$ of V spans (respectively, is independent in, respectively, is a basis of) V if and only if $\{\widetilde{v_1}, \ldots, \widetilde{v_k}\}$ has the corresponding property with respect to $\widetilde{V}$. Conclude that $\dim V = \dim \widetilde{V}$.

Proposition. *Suppose V is a complex vector space with Hermitian inner-product. The function*

$$\phi : \widetilde{V} \to V'; v \mapsto \phi_v$$

defined by

$$\phi_{\tilde{v}}(u) := \langle u, v\rangle$$

(i.e., $\phi_{\tilde{v}} = \langle \cdot, v\rangle$) is a bijective linear map.

Proof. We have $\phi_{c\tilde{v}}(u) = \langle u, \overline{c}v\rangle = c\langle u, v\rangle = c\phi_{\tilde{v}}(u)$. The rest of the proof is just as in the real case. /////

Comments. There are several other ways to approach the issue in this section. See Knapp's Theorem 3.12 (page 98) for an alternate approach. I will also supply copies from the nice book by Axler, *Linear Algebra Done Right*. Lang uses anti-modules; see his *Algebra*, page 531.

From the propositions we can conclude:

Theorem. *If $\ell : V \to \mathbb{F}$ is any linear functional on V, then there is a unique $w_\ell \in V$ such that*

$$\ell(v) = \langle v, w_\ell\rangle \qquad \text{for all } v \in V. \qquad /////$$

Homework: Exercises above. Hand in P. 111–112: 3, 4, 6, 8. Also, try: P.114: 24–28.
