M7210 Lecture 14 Monday September 24, 2012

Adjoints and the Spectral Theorem

I will be describing the highlights of section III.2 of Knapp. As previously, we assume

$$\boxed{V \text{ is a finite-dimensional inner-product space over } \mathbb{F} = \mathbb{R} \text{ or } \mathbb{F} = \mathbb{C}.}$$

If $L : V \to V$ is a linear map, we may form the function

$$(u, v) \mapsto \langle L(u), v\rangle : V \times V \to \mathbb{F}.$$

This gives us a map from $\mathrm{Lin}(V, V)$, the vector-space of all linear maps from V to V, to $\mathrm{Sesq}(V \times V, \mathbb{F})$, the vector space of all sesquilinear maps from $V \times V$ to $\mathbb{F}$:

$$L \mapsto \langle L(\cdot), \cdot\rangle : \mathrm{Lin}(V, V) \to \mathrm{Sesq}(V \times V, \mathbb{F}).$$

This map is injective, because if L is not the zero map, then $L(u) \neq 0$ for some $u \in V$, and hence $\langle L(\cdot), \cdot\rangle$ is non-zero, as we can see by evaluating it at $\big(u, L(u)\big)$.

Proposition. *Let $L : V \to V$ be a linear map. For each $u \in V$, there is a unique $L^*(u) \in V$ such that*

$$\langle L(v), u\rangle = \langle v, L^*(u)\rangle \qquad \textit{for all } v \in V.$$

Moreover, for any $u, u_1, u_2 \in V$, $L^*(cu) = cL^*(u)$ and $L^*(u_1 + u_2) = L^*(u_1) + L^*(u_2)$.

Proof. Existence and uniqueness follows from the last theorem in Lecture 13. For any fixed u, we have for all $v \in V$: $\langle v, L^*(cu)\rangle = \langle L(v), cu\rangle = \overline{c}\langle L(v), u\rangle = \langle v, cL^*(u)\rangle$. Thus, $L^*(cu) = cL^*(u)$ (by the uniqueness part of the last theorem in the last lecture. Additivity follows similarly. /////

Definition. If $L = L^*$, then L is said to be *self-adjoint*.

Comment. This definition has the advantage that it is independent of any basis, but it has the disadvantage that it makes the adjoint seem mysteriously abstract. In fact, if A is the matrix of L relative to selected orthonormal bases on the domain and codomain (each a copy of V), then the the matrix of L^* relative to these bases is the congugate transpose of A. So, you can easily obtain notation for L^* if you need to do concrete computations. See page 99-100 of Knapp for details.

Example. (Cf. Proposition 3.16, page 100.) Suppose U is a subspace of V. Let

$$U^\perp = \{v \in V \mid v \perp u \text{ for all } u \in U\}$$

be the orthogonal complement of U. Then, each element of $v \in V$ can be written uniquely as a sum $\pi_U(v) + \pi_{U^\perp}(v)$, with $\pi_U(v) \in U$ and $\pi_{U^\perp}(v) \in U^\perp$. I assert that π_U is self-adjoint. Indeed, for any $v, w \in V$:

$$\langle v, \pi_U^*(w)\rangle = \langle \pi_U(v), w\rangle = \langle \pi_U(v), \pi_U(w)\rangle = \langle v, \pi_U(w)\rangle.$$

Caution. There are product decompositions of V that are not "orthogonal". Indeed, if U and W are any non-trivial subspaces of V with $U \cap W = \{0\}$ and $\dim U + \dim W = \dim V$, then $V = U \oplus W$ as a vector-space sum/product. As a matter of fact, given specific U and W with these properties, there are many different mappings that could serve as projections $\pi_U : V \to U$

and $\pi_W : V \to W$. Even when $U \perp W$ with respect to some inner product, there are projections that are not the orthogonal ones. It is very important to distinguish clearly between constructions that respect only the vector-space structure and constructions that respect the inner product. In the example above, the projection must respect the inner product.

Exercise. Call V, with inner product $\langle \cdot, \cdot \rangle$ the *orthogonal direct sum of* V_1 *and* V_2 if:

- $V = V_1 \oplus V_2$ as vector spaces,
- V_1, and V_2 each has its own inner product, $\langle \cdot, \cdot \rangle_1$, $\langle \cdot, \cdot \rangle_2$ respectively, and
- $\big\langle (v_1, v_2), (w_1, w_2) \big\rangle = \langle v_1, w_1 \rangle_1 + \langle v_2, w_2 \rangle_2$.

(a) Suppose $V = S \oplus S^\perp$ is an orthogonal direct sum. Also, suppose $L : V \to V$ is self-adjoint, and $L(S) \subseteq S$. Show that the restriction of L to S is self-adjoint.

(b) Suppose V is the orthogonal direct sum of V_1 and V_2. Find surjective linear functions $\beta_i : V \to V_i$ so that V is the vector-space product with projections β_1, β_2, but

$$\big\langle v, w \big\rangle \neq \big\langle \beta_1(v), \beta_1(w) \big\rangle_1 + \big\langle \beta_2(v), \beta_2(w) \big\rangle_2 .$$

Proposition. (3.17, part 1) *Suppose $L : V \to V$ is self-adjoint. Then*
(a) $\langle L(v), v \rangle \in \mathbb{R}$ for every $v \in V$.
(b) Every eigenvalue of L is real.

Proof. a) If $L = L^*$, then

$$\langle L(v), v \rangle = \langle v, L(v) \rangle = \overline{\langle L(v), v \rangle}.$$

b) If $L(v) = \lambda v$, then

$$\lambda \langle v, v \rangle = \langle \lambda v, v \rangle = \overline{\langle \lambda v, v \rangle} = \overline{\lambda} \langle v, v \rangle.$$

Lemma. (3.20) *If $L : V \to V$ is self-adjoint, then*
(a) eigenvectors for distinct eigenvalues of L are orthogonal, and
(b) for any subspace $S \subseteq V$, if $L(S) \subseteq S$, then $L(S^\perp) \subseteq S^\perp$.

Proof. (a) Suppose v_1 and v_2 are eigenvectors with distinct eigenvalues $\lambda_1 \neq \lambda_2$. By 3.17, part 1, $\lambda_1, \lambda_2 \in \mathbb{R}$. Then

$$(\lambda_1 - \lambda_2) \langle v_1, v_2 \rangle = \langle \lambda_1 v_1, v_2 \rangle - \langle v_1, \lambda_1 v_2 \rangle = \langle L(v_1), v_2 \rangle - \langle v_1, L(v_2) \rangle = 0.$$

Thus $\langle v_1, v_2 \rangle = 0$. For part (b), if $L(S) \subseteq S$, then for any $s \in S$ and $s^\perp \in S^\perp$,

$$0 = \langle L(s), s^\perp \rangle = \langle s, L(s^\perp) \rangle.$$

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Theorem. ($\mathbb{F} = \mathbb{C}$ case of the Spectral Theorem, 3.21) *Suppose V is a complex inner-product space and $L : V \to V$ is self-adjoint. Then V has an orthonormal basis consisting of eigenvectors of L.*

Proof. Let V_λ be the eigenspace for L with eigenvalue λ. Then the characteristic polynomial of L has at least one root, λ_1, and so we have a nontrivial orthogonal decomposition $V = V_{\lambda_1} \oplus V_{\lambda_1}^\perp$. Moreover, $L(V_{\lambda_1}) \subseteq V_{\lambda_1}$, and $L(V_{\lambda_1}^\perp) \subseteq V_{\lambda_1}^\perp$. Take any orthogonal basis for V_{λ_1}. By part (a) of the exercise above, $V_{\lambda_1}^\perp$ is an inner product space and the restriction of L to $V_{\lambda_1}^\perp$ is self-adjoint. So by induction (on dimension), we have a basis for $V_{\lambda_1}^\perp$ satisfying the requirements of the theorem, and combined with the basis for V_{λ_1}, we are done. /////

Theorem 3.21, with $\mathbb{F} = \mathbb{R}$ can be approached two ways (at least). In Knapp, we deduce it from the complex case. Another approach is to show that a self-adjoint operator on a real space has an eigenvector, then imitate the above.