M7210 Lecture 17 Monday October 1, 2012

Group actions.

We ended the last lecture by introducing the set of cosets $G/H := \{\, gh \mid g \in G \,\}$ of and arbitrary subgroup H of a group G. When H is normal, G/H has the structure of a group. The main theme of this lecture is to examine the structure that G/H possesses when H is not normal, provide an axiomatic characterization (transitive G-set) and give some applications.

Definition. Let G be a group. A *G-set* is a set X equipped with a mapping

$$G \times X \to X; (g, x) \mapsto gx$$

called the *action of G on X*, that satisfies the following axioms:
i) for all $x \in X$, $ex = x$;
ii) for all $f, g \in G$ and all $x \in X$, $f(gx) = (fg)x$.

An action is a function $\alpha : G \times X \to X$. Often, we do not name it explicitly, but we will use the name α in this paragraph and below when clarity is needed. Let us adopt the notation $\alpha_g(x)$ for the image of (g, x). If we select an element g of G and hold it fixed, then α_g is a function from X to X. Axiom *i*) says that $\alpha_e = \mathrm{id}_X$ and axiom *ii*) says that $\alpha_f \alpha_g = \alpha_{fg}$. The two axioms together imply that α_g is the inverse function of $\alpha_{g^{-1}}$. Therefore, since G is a group, each function $\alpha_g : X \to X$ is a bijection, and the map $g \mapsto \alpha_g : G \to \mathrm{Bij}(X, X)$ is a homomorphism. Conversely, if $g \mapsto \alpha_g : G \to \mathrm{Bij}(X, X)$ is any homomorphism of groups, then $(g, x) \mapsto \alpha_g(x)$ is an action of G on X.

Examples.

1. G itself is a G-set, where the action $G \times G \to G$ is the group operation. We referred to this action in the proof of Cayley's Theorem, where we used a different notation in order to highlight the distinction between the notions of group *operation* and group *action*.

2. Let H be a subgroup of G. Then G acts on G/H via

$$(f, gH) \mapsto fgH = \alpha_f(gH).$$

Since we have defined $\alpha_f(gH)$ by reference to one of many possible names for the coset gH, we need to show that it is well-defined. But

$$\begin{aligned} g_0H = g_1H &\Leftrightarrow g_1^{-1}g_0 \in H \\ &\Leftrightarrow g_1^{-1}f^{-1}fg_0 \in H \\ &\Leftrightarrow (fg_1)^{-1}(fg_0) \in H \\ &\Leftrightarrow fg_1H = fg_0H. \end{aligned}$$

3. A set of the form Hg is called a right coset. The collection of all right cosets of H in G is denoted $H \setminus G$. We have $Hg = Hf \Leftrightarrow H = Hfg^{-1} \Leftrightarrow fg^{-1} \in H$. G acts on $H \setminus G$ by

$$(f, Hg) \mapsto Hgf^{-1} = \alpha_f(Hg).$$

This is well-defined, since $Hg_1 = Hg_2 \Leftrightarrow g_1g_2^{-1} \in H \Leftrightarrow g_1f^{-1}fg_2^{-1} \in H \Leftrightarrow Hg_1f^{-1} = Hg_2f^{-1}$. It is an action, since $\alpha_k(\alpha_f(Hg) = Hgf^{-1}k^{-1} = Hg(kf)^{-1} = \alpha_{kf}(Hg)$.

4. G acts on $X = G$ by conjugation as follows:

$$(g, x) \mapsto gxg^{-1}.$$

5. The group E Euclidean symmetries (a.k.a. isometries) of the complex plane is the subgroup of the group of all bijections of $\mathbb{C}$ with itself generated by D_∞ and $\mathbb{C}$ itself acting by translations. Each element of E is a function f of the form $f(z) = az + b$ or $f(z) = a\overline{z} + b$, where $a, b \in \mathbb{C}$ and $|a| = 1$. The action is $(f, z) \mapsto f(z)$.

6. For any field $\mathbb{F}$, the group $G = \mathrm{GL}(n, \mathbb{F})$ of all invertible $n \times n$ matrices acts on $X = \mathbb{F}^n$ by matrix multiplication: $(A, v) \mapsto Av$.

7. The group $G = \mathrm{GL}(2, \mathbb{C})$ acts on $X = \mathbb{C} \cup \{\infty\}$ by linear fractional transformations. The subgroup $\mathrm{SL}(2, \mathbb{C}) \subset G$ of matrices with determinant 1 acts on $\{\, x + iy \mid x, y \in \mathbb{R},\ 0 < y \,\}$. For more detail about these interesting and important examples, read pages 159-160.

Definitions. Let $G \times X \to X$ be a group action and let $x \in X$.

i) $G_x := \{\, g \in G \mid gx = x \,\}$ is called the *isotropy group of* x, or *stabilizer* of x.

ii) $Gx := \{\, gx \mid g \in G \,\}$ is called the *orbit of* x.

An action is said to be *transitive* if it has only one orbit.

Note that X is a disjoint union of orbits. In other words, "belonging to the same orbit" is an equivalence relation on X.

Lemma. *If $y = hx$ for some $h \in G$. Then $G_y = hG_xh^{-1}$. Thus, if x and y are is the same G-orbit, then G_y is a conjugate of G_x.*

Proof. $G_y = \{\, g \in G \mid ghx = hx \,\} = \{\, g \in G \mid h^{-1}gh \in G_x \,\} = hG_xh^{-1}$.

Structure of G-sets. Suppose X and Y are G-sets. A *G-set morphism from X to Y* is a function $\phi : X \to Y$ such that $\phi(gx) = g\phi(x)$ for all $x \in X$. A G-set isomorphism is a bijective G-set morphism.

Note that the that ϕ is a G-set isomorphism if and only if it has two-sided inverse as a G-set morphism. For suppose $\phi : X \to Y$ is a bijective G-set morphism. Let ϕ^{-1} be its set-theoretic inverse. If $y \in Y$, then $y = \phi(x)$ for a unique $x \in X$, and $\phi^{-1}(gy) = \phi^{-1}(g\phi(x)) = \phi^{-1}(\phi(gx)) = gx = g\phi^{-1}(y)$. Thus, ϕ^{-1} is a G-set morphism.

Proposition. *Suppose X is a transitive G-set. Let x be a fixed element of X and let H be its isotropy group. Define*

$$\phi : G/H \to X; \phi(gH) := gx.$$

Then ϕ is a well-defined G-set isomorphism.

Proof. The function is well-defined, for suppose $g_1H = g_2H$; then $g_2^{-1}g_1 \in H$, so $g_2^{-1}g_1x = x$, so $g_1x = g_2x$. It is a G-set morphism, for $\phi(fgH) = (fg)x = f(gx) = f\phi(gH)$. To see it is injective, suppose $\phi(g_1H) \neq \phi(g_2H)$; then $g_2^{-1}g_1x \neq x$ so $g_2^{-1}g_1 \notin H$ so $g_1H \neq g_2H$. It is surjective because X is transitive, so each element of X is equal to $gx = \phi(gH)$ for some $g \in G$ /////

Corollary. *(Counting orbits.) If G is a finite group, X is a G-set and $x \in X$, then*

$$|G| = |Gp|\,|G_p|.$$

Exercise. Suppose X is a G-set. Let $\sim$ be an equivalence relation on X. Say that $\sim$ is *compatible with the G-action* if

$$\text{for all } x, x' \in X \text{ and all } g \in G,\ x \sim x' \ \Rightarrow\ gx \sim gx'.$$

Show that the G-compatible equivalence relations on G/H are in one-to-one correspondence with the subgroups $K \subseteq G$ that contain H.

Proposition. *If G is a finite group let p is the smallest prime dividing the order of G. Then any subgroup of index p is normal.*

Proof. . Let H be a subgroup of index p. Let G act on G/H by left translation, and restrict this to an action of H on G/H. Then $\{H\}$ is a single orbit, and the remaining $p-1$ cosets of H form a union of orbits. Now, the number of elements in any orbit of H is a divisor of the order of H, but all divisors of $|H|$ are $\geq p$. Therefore, all the orbits of H are singletons. This means $hgH = gH$ for all $g \in G$, so $g^{-1}Hg \subseteq H$ for all $g \in G$. /////

Will continue with **Action by conjugation, centralizers and class equation.**