**M7210 Lecture 18** Wednesday October 3, 2012

Today, we have three goals.

1. Apply the theory of group actions to conjugation and derive the "class equation".
2. Apply the class equation to $p$-groups. We will use the Isomorphism Theorem, which we have not yet proved. Actually, we will use the ideas in it, so we will not need to state and prove it but just really understand the meaning of it.
3. We'll return to products and (time permitting) prove a useful criterion for recognizing products. This is in preparation for semidirect products, which we'll get to on Friday.

Recall the following important points from the last lecture:

- A *$G$-set* is a set $X$ equipped with a mapping $G \times X \to X; (g, x) \mapsto gx$ such that: *i*) for all $x \in X$, $ex = x$, and *ii*) for all $f, g \in G$ and all $x \in X$, $f(gx) = (fg)x$.
- $G_x := \{ g \in G \mid gx = x \}$ is called the *isotropy group of $x$*, or *stabilizer of $x$*.
- $Gx := \{ gx \mid g \in G \}$ is called the *orbit of $x$*.

We showed that if $G$ is a finite group and $X$ is a $G$-set, then for any $x \in X$,

$$|G| = |Gx|\,|G_x|. \tag{1}$$

In other words, the cardinality of any orbit times the cardinality of the stabilizer of any element of that orbit is the cardinality of $G$.

**Action by conjugation, centralizers and class equation.**

$G$ acts on $X = G$ by conjugation according to the rule: $(g, x) \mapsto gxg^{-1}$. Notice that we must have $g^{-1}$ on the right in order to satisfy condition *ii*) of the definition of $G$-set. We will assume that we are dealing with this action and that $G$ is a finite group.

Let $x \in G$. Then $g \in G$ is in the stabilizer of $x$ if and only if $gxg^{-1} = x$ if and only if $gx = xg$, i.e., $x$ commutes with $g$. The set of all $g \in G$ that satisfy this condition is called the *centralizer of $x$* and is denoted $Z_G(x)$. The intersection of all centralizers is called the *center of $G$* and is denoted $Z_G$. It contains the elements of $G$ that commute with all $x \in G$.

**Exercise 1.** Show that if $G/Z_G$ is cyclic, then $G$ is abelian. (Hint: Suppose $G/Z_G = \langle aZ_G \rangle$. If $g, h \in G$ then $g = a^i z_1$ and $h = a^j z_2$ for some $i, j \in \mathbb{Z}$ and $z_1, z_2 \in Z_G$.)

**Exercise 2.** Find a non-abelian group such that $G/Z_G$ is abelian.

**Exercise 3.** In the symmetric group $S_6$, what is the centralizer of (12)? Of (123)? Of (1234)? Of (12)(34)? Of (12345)? Of (123)(45)?

The orbit of $x$ under conjugation is $\{ gxg^{-1} \mid g \in G \}$. This is called the *congugacy class of $x$* and is denoted $\mathcal{C}\ell(x)$. From Equation 1:

$$|G| = |\mathcal{C}\ell(x)|\,|Z_G(x)| \quad \text{or, equivalently} \quad |\mathcal{C}\ell(x)| = \frac{|G|}{|Z_G(x)|}. \tag{2}$$

If the set $\{ x, y, \ldots, z \} \subset G$ contains one element from each congugacy class, then:

$$|G| = |\mathcal{C}\ell(x)| + |\mathcal{C}\ell(y)| + \ldots + |\mathcal{C}\ell(z)|.$$

Now $|\mathcal{C}\ell(x)| = 1$ if and only if $x \in Z_G$, so this can be written:

$$|G| = |Z_G| + \sum_{x \in R} \frac{|G|}{|Z_G(x)|}, \qquad \text{(class equation)}$$

where $R$ contains one representative from each *non-singleton* conjugacy class of $G$.

**Some results on $p$-groups.**

Let $p$ be a positive prime integer. A group with $p^k$ element is called a $p$-group. We will prove three results about $p$-groups.

**4.38.** *Every $p$-group has non-tivial center.*

*Proof*. In the class equation, the sum part is divisible by $p$ while $|Z_G| \geq 1$. /////

**4.39.** *Any group with $p^2$ elements is abelian.*

*Proof*. We know $|Z_G| \geq p$. Suppose $|Z_G| = p$. Select $x \in G \setminus Z_G$. Then $Z_G(x)$ contains $Z_G$ as well as $x$, so $|Z_G(x)| = p^2$. But this implies $x \in Z_G$—contradiction. (An alternate proof uses Exercise 1.) /////

**4.40.** *Any group with $p^n$ elements has normal subgroups $G_k$ of order $p^k$ for each $k = 0, 1, \ldots, n$ such that $G_k \subseteq G_{k+1}$ for each $k$ such that $0 \leq k < n$.*

*Proof*. This is clearly true if $n = 0, 1, 2$ by previous results. Suppose the theorem is true for any $p$-group with strictly less than $p^n$ elements, and suppose $G$ has $p^n$ elements. Let $G_0 = \{0\}$ and let $H$ be any a nontrivial group of order $p$ within $Z_G$. Then $H$ is normal in $G$. By the inductive hypothesis, $\overline{G} := G/H$ has normal subgroups $\overline{G_0} \subseteq \overline{G_1} \subseteq \cdots \subseteq \overline{G_{n-1}}$, with $|\overline{G_i}| = p^i$. Let $\pi : G \to \overline{G}; g \mapsto gH$, and let $G_{i+1} := \pi^{-1}\overline{G_i}$ for $i = 0, \ldots, n-1$ (so $G_1 = H$). These groups have the required properties. /////

*Remark on proof.* $G_{i+1}$ is normal because it is the kernel of the composite homomorphism

$$g \mapsto \pi(g) \mapsto \pi(g)\overline{G_i}.$$

**Read** about congugacy classes in $S_n$ on pages 165-6.

This is as far as we got. Next time, we will look at semidirect products.