M7210 Lecture 20 Monday October 8, 2012

Categories and Functors

Much of modern mathematics can be understood as the study of sets with structure. Groups, rings and modules are obvious examples—all are sets equipped with operations that satisfy certain axioms. Topological spaces are also sets equipped with structure, but in this case the structure is not described by means of operations but by some notion of closeness. In analysis, the objects have both algebraic and topological structure. In most mathematical work, one is not concerned with a single structure, but rather with many structures of a given type and the relations between various individual structures of that type that can be described by means of functions that preserve that structure. Group theory provides a good example. Even if we set out to understand the properties of a single group, we often find ourselves looking at the subgroups and homomorphic images of that group.

Category theory provides a useful way of thinking about the relationships that hold between the structures of a given kind and of comparing the relationships among the representatives of one kind (e.g., topological spaces) with the relationships among the representatives of another (e.g. modules). Category theory takes us even further, allowing us to generalize the notion of structure beyond the set-theoretic universe. There are categories whose objects are not structured sets, and whose properties are understood entirely in terms of the morphisms that link them to other objects. This may make category theory seem mysterious and esoteric, yet one of its greatest contributions to mathematics is quite mundane and practical: it is a compact and efficient language for expressing the kinds of abstractions that are routine in modern mathematics.

But I have offered enough appetizers! Let us proceed to the main course!

Definition. A *category* consists of a collection of *objects* and a collection of *morphisms* satisfying the following axioms:

1) Every morphism f is associated with two objects, called respectively the *domain* and the *codomain* of f. If f has domain A and codomain B, we write $f : A \to B$. For any two objects, A and B, the collection of all morphisms with domain A and codomain B is a set, denoted $\mathrm{morph}(A, B)$. $\mathrm{morph}(A, B)$ and $\mathrm{morph}(C, D)$ are disjoint, unless $A = C$ and $B = D$.
2) If $f : A \to B$ (i.e., $f \in \mathrm{morph}(A, B)$) and $g : B \to C$, then there is a morphism $gf : A \to C$. For every object A, there is a morphism $\mathrm{id}_A : A \to A$ called the *identity on* A.
3) Composition of morphisms is associative: $h(gf) = (hg)f$ whenever the compositions exist. The identity morphism on any object is a left and right identity: $f\,\mathrm{id}_A = f$ and $\mathrm{id}_B\, f = f$ for all objects A and B and morphisms f, provided the compositions exist.

Routine examples.

1) **Set** is the category of sets and functions. The objects of this category are the sets and the morphisms are the functions between sets.
2) The category **Group** has as its objects all groups. Its morphisms are the group homomorphisms. **Ab** is the category of abelian groups. Its objects are the abelian groups, and for any abelian groups A and B, $\mathrm{morph}_{\mathbf{Ab}}(A, B) = \mathrm{morph}_{\mathbf{Group}}(A, B)$.
3) **Rng** is the category of rings and ring homomorphisms. In Knapp, rings are not assumed to possess multiplicative identity. The collection os all rings with identity together with the identity-preserving ring homomorphisms between them forms a different category, called **Ring**. (This naming convention was suggested by N. Jacobson. Not everyone uses it.)
4) The vector spaces over $\mathbb{R}$ and the linear maps between them form a category. The vector spaces over $\mathbb{C}$ and the linear maps between them form a different category, and indeed, for each different field $\mathbb{F}$ there is a different category of vector spaces over that field.
5) Fix a group G. The category of G-sets, denoted G-**Set** has G-sets as objects and G-set morphisms (defined in Lecture 17) as morphisms.

Definition. A category $\mathbf{D}$ is a *subcategory* of the category $\mathbf{C}$ if the objects of $\mathbf{D}$ form a subclass of the objects of $\mathbf{C}$ and for any objects A and B of $\mathbf{D}$, $\mathrm{morph}_{\mathbf{D}}(A, B) \subseteq \mathrm{morph}_{\mathbf{C}}(A, B)$. We require that identity morphisms and compositions agree in the two categories. $\mathbf{D}$ is said to be *full* in $\mathbf{C}$ if $\mathrm{morph}_{\mathbf{D}}(A, B) = \mathrm{morph}_{\mathbf{C}}(A, B)$ for all objects A, B of $\mathbf{D}$.

Examples.

1) **Ab** is a full subcategory of **Group**.
2) **Group** is not a subcategory of **Set** because the objects are different. A group is a set equipped with operations—not just a set. The relationship between **Group** and **Set** is described by functors; see below.

Two surprising—but important—examples.

1) Let G be a group. Then G itself is a category. It has one object and has one morphism for each element of G.
2) Let X be a partially ordered set—i.e., a set equipped with a reflexive, antisymmetic, transitive relation $\leq$. Then we may view X as a category that has one object for each $x \in X$. We let $\mathrm{morph}(x, y)$ contain a single element if $x \leq y$. It is empty otherwise.

In trying to absorb these two examples, you may wonder what the single object in the category G is. But this is not important. Our attention is on the features of the category that can be detected by the relations among its objects and morphisms, and we studiously ignore the issue of what the objects and morphisms might be "internally." Similarly, if we regard $\mathbb{Z}$ (the integers) as the category determined by its structure as ordered set, then we will say that there is exactly one morphism from 1 to 2, but we have no interest in any information about it that cannot be stated by reference to the other objects and morphisms in the category. The morphism from 1 to 2 may be composed with the morphism from 2 to 17 to obtain the unique morphism from 1 to 17. The morphism from 1 to 1 is id_1. If I take the composition of id_1 followed by the unique morphism from 1 to 2, the result is the unique morphism from 1 to 2. Etc.

Categorical concepts

Some special morphisms

Definition. Let $\mathbf{C}$ be a category, let A and B be objects of $\mathbf{C}$ and let $f : A \to B$.

1) If there is $g : B \to A$ such that $gf = \mathrm{id}_A$ and $fg = \mathrm{id}_B$, then f is said to be an *isomorphism*. If $\mathrm{morph}(A, B)$ contains an isomorphism, then we say that A and B are *isomorphic* or *equivalent*.
2) f is said to be *monic* if it is left cancelable, i.e., for all $g_1, g_2 : C \to A$, if $fg_1 = fg_2$, then $g_1 = g_2$. f is said to be *epic* if it is right cancelable, i.e., for all $g_1, g_2 : B \to C$, if $g_1 f = g_2 f$, then $g_1 = g_2$.

Examples.

1) A morphism of sets is an isomorphism if and only if it is bijective. The same thing is true in **Group**, **Ring** and G-**Set**.
2) In the category determined by a group G, every morphism is an isomorphism. In the category determined by a partially-ordered set, the only isomorphisms are the identity morphisms.
3) A morphism in **Group** or **Ring** is monic if an only if it is injective.

Proposition.

(a) *A morphism of groups is epic if and only if it is surjective.*
(b) *The embedding of $\mathbb{Z}$ in $\mathbb{Q}$ is epic in* **Ring**.

Proof. (a) [Linderholm, C., "A Group Epimorphism is Surjective," *Am. Math. Monthly* 77, 176-7.] Let $f : G \to H$ be a homomorphism of groups that is not surjective. Let $A = f(G) \subset H$. We need to find a group K and two homomorphisms $\alpha, \beta : H \to K$ such that $\alpha \neq \beta$ and $\alpha f = \beta f$ (i.e., $\alpha = \beta$ on A). The strategy will be as follows. We have the standard action of H on H/A. We described this in Lecture 17. It is a homomorphism from H to $\mathrm{Bij}(H/A)$. We are going to augment the set H/A to create a larger set, and then extend the action of H on H/A to this larger set to create α. Then we create β by a slight modification of α. Let $S = H/A \cup \{X\}$, where X is a token object that is not a left coset of A in H. Let $K = \mathrm{Bij}(S)$. For each $h \in H$, let $\alpha_h \in K$ be defined by $\alpha_h(h_1A) := hh_1A$ and $\alpha_h(X) := X$. Thus, α is an extension of the natural action of H on H/A. Let $\sigma \in K$ be defined by $\sigma(X) := A$, $\sigma(A) := X$ and $\sigma(h_1A) := h_1A$ if $h_1 \in H$ but $h_1 \notin A$. Let $\beta_h := \sigma\alpha_h\sigma$. One can easily check that $\beta : H \to K; h \mapsto \beta_h$ is a homomorphism. We now show that $\alpha_a = \beta_a$ for all $a \in A$. Suppose $a \in A$. If $h_1A \neq A$, then $\alpha_a(h_1A) = ah_1A$. Note that $ah_1A \neq A$. In the meantime, $\beta_a(h_1A) = \sigma\Big(\alpha_a\big(\sigma(h_1A)\big)\Big) = \sigma\Big(\alpha_a\big(h_1A\big)\Big) = \sigma\Big(ah_1A\Big) = ah_1A$. Now, α_a and β_a also agree on A: $\alpha_a(A) = A = \beta_a(A)$. Thus $\alpha_a = \beta_a$ for all $a \in A$. On the other hand, if $h \notin A$, then $\alpha_h(h^{-1}A) = A$, but

$$\begin{aligned}\beta_h(h^{-1}A) &= \sigma\Big(\alpha_h\big(\sigma(h^{-1}A)\big)\Big)\\ &= \sigma\big(\alpha_h(h^{-1}A)\big)\\ &= \sigma(A) = X.\end{aligned}$$

This completes the proof of (a). Below is more detail about the actions, which you might find useful in understanding the proof. We are assuming here that $h \notin A$.

$$\alpha_h : h^{-1}A \mapsto A \mapsto hA;\ X \mapsto X$$

$$\beta_h : h^{-1}A \mapsto X \mapsto hA,\ A \mapsto A$$

(b) Suppose $g_1, g_2 : \mathbb{Q} \to R$ are two (identity-preserving!) ring homomorphisms that agree on $\mathbb{Z}$. (R is an arbitrary ring.) We need to show that $g_1(m/n) = g_2(m/n)$ for all $m, n \in \mathbb{Z}$, $n \neq 0$. In case $R = \{0\}$, this is obvious. So, suppose R is not the zero ring. Then,

$$ng_1(m/n) = g_1(m) = g_2(m) = ng_2(m/n).$$

We need to show that this implies $g_1(m/n) = g_2(m/n)$. Suppose x and y are any elements of R and $nx = ny$. Then $n(x-y) = 0$. But then, $0 = g_1(1/n) \cdot 0 = g_1(1/n)\big(n(x-y)\big) = \big(g_1(1/n)n\big)(x-y) = g_1(1)(x-y) = (x-y)$, so $x = y$.

Some special objects

Definition. Let $\mathbf{C}$ be a category. An object A of $\mathbf{C}$ is said to be *initial* if for every object X of $\mathbf{C}$ there is a unique morphism $f : A \to X$. An object Z of $\mathbf{C}$ is said to be *final* if for every object X of $\mathbf{C}$ there is a unique morphism $f : X \to Z$.

Examples. In **Set**, the empty set is initial, and any one-element set is final. In **Group**, the one-element group $\{e\}$ is both initial and final. In the category of rings with multiplicative identity, $\mathbb{Z}$ is initial and the zero-ring $\{0\}$ is final. The category G has no initial or final element, unless G is the trivial group. A partially-ordered set has an initial (final) object if and only if it has a least (greatest) element.

Definition. Let $\{\, A_\lambda \mid \lambda \in \Lambda \,\}$ be a collection of objects of $\mathbf{C}$ indexed by a set Λ. The *categorical product* of the A_λ is an object P in $\mathbf{C}$ and a collection of morphisms $\{\, \pi_\lambda : P \to A_\lambda \mid \lambda \in \Lambda \,\}$ such that for any object T and morphisms $\alpha_\lambda : T \to A_\lambda$, there is a unique morphism $\alpha : T \to P$ such that $\alpha_\lambda = \pi_\lambda\alpha$ for all λ.

The object P when it exists, is unique up to isomorphism. It is denoted $\prod_{\lambda\in\Lambda} A_\lambda$.

Definition. Let $\{\, A_\lambda \mid \lambda \in \Lambda \,\}$ be a collection of objects of $\mathbf{C}$ indexed by a set Λ. The *categorical sum* of the A_λ is an object S in $\mathbf{C}$ and a collection of morphisms $\{\, \iota_\lambda : A_\lambda \to S \mid \lambda \in \Lambda \,\}$ such that for any object T and morphisms $\alpha_\lambda : A_\lambda \to T$, there is a unique morphism $\alpha : S \to T$ such that $\alpha_\lambda = \alpha\iota_\lambda$ for all λ.

The object S when it exists, is unique up to isomorphism. It is denoted $\coprod_{\lambda \in \Lambda} A_\lambda$. (In some categories, other notations are traditional.)

Previous discussions have shown that in **Set**, $\prod_{\lambda \in \Lambda} A_\lambda$ is the usual cartesian product and in **Group**, $\prod_{\lambda \in \Lambda} A_\lambda$ is the cartesian product equipped with component-wise operations. In **Set**, $\coprod_{\lambda \in \Lambda} A_\lambda$ is the disjoint union. (Here is an instance where a different notation is traditional: a union-symbol with a dot inside it.) In **Group**, $\coprod_{\lambda \in \Lambda} A_\lambda$ exists but the description is complicated; see the section VII. 3. Free Products, pp. 319-326 of Knapp. On the other hand, in **Ab**, $\coprod_{\lambda \in \Lambda} A_\lambda$ is the subgroup of $\prod_{\lambda \in \Lambda} A_\lambda$ that is generated by the images of the A_λ. If Λ is finite, then the two objects are equal.

Functors

A morphism of categories is called a functor. To be precise:

Definition. Let **C** and **D** be categories. A *functor* from **C** to **D** is a rule that assigns to each object A of **C** and object $F(A)$ of **D** and to each morphism $f : A \to B$ of **C** a morphism $F(f) : F(A) \to F(B)$. This assignment must preserve identities and compositions whenever they exist:

$$F(\mathrm{id}_A) = \mathrm{id}_{F(A)}, \qquad \text{and} \qquad F(gf) = F(g)F(f).$$