Lecture 2. Algebraic constructions and categories

We define some categories and display some constructions that will play a role later.

M-sets

Definition. Let M be a monoid. A *left M-set* is a set X equipped with a function $M \times X \to X; (m, x) \mapsto mx$ such that $ex = x$ and $m(m'x) = (m * m')x$ for all $m, m' \in M$ and $x \in X$. A right M-set is a set X equipped with a function $X \times M \to X; (x, m) \mapsto xm$ such that $xe = x$ and $(xm)m' = x(m * m')$ for all $m, m' \in M$ and $x \in X$.

2.1. Facts.

1. A congruence C on M is both a left and right M-set, with the actions defined as follows:
$$M \times C \to C; (m, (m', m'')) \mapsto (m * m', m * m'')$$
$$C \times M \to C; ((m', m''), m) \mapsto (m' * x, m'' * m).$$
2. An equivalence relation E on M that is both a right and left M-set under the action described above is a congruence.

2.2. Definition. A left M-set morphism is a function ϕ between left M-sets that satisfies $m\phi(x) = \phi(mx)$. A sub-left-M-set of an M-set is a subset that is closed under the action of M. The product of an I-indexed family of M-sets $\{ X_i \mid i \in I \}$ is the set-theoretic product, with the component-wise action: $(my)_i := m(y_i)$, for $y \in \prod_{i\in I} X_i$. A left M-set congruence on X is an equivalence relation on X that is also sub-left-M-set of $X \times X$. The set of equivalence classes of a congruence C on S is naturally an M-set. It is denoted X/C and is called the quotient of X by C.

2.3. Exercise. Let I be a set. Show that I is a left M-set under the trivial action $(m, i) \mapsto i$. Suppose that X is a left M-set and $\phi : I \to X$ is any function. Show that there is a unique left M-set morphism $\overline{\phi} : M \times I \to X$ such that $\overline{\phi}(m, i) = m\phi(i)$. (In other words, $M \times I$ is the free left M-set on I.)

M-modules

Definition. Let M be a monoid. A *left M-module* is an abelian group A equipped with a function $M \times A \to A; (m, a) \mapsto ma$ such that $ea = a$ and $m(m'a) = (m * m')a$ and $m(a + b) = ma + mb$ for all $m, m' \in M$ and $a, b \in A$. Right M-module is defined analogously.

From now on, unless otherwise stated, "M-module" means left M-module. A function $\phi : A \to A'$ between M-modules is a *morphism of M-modules* if $\phi(m_1a_1 + m_2a_2) = m_1\phi(a_1) + m_2\phi(a_2)$ for all $m_1, m_2 \in M$ and $a_1, a_2 \in A$. A *sub-M-module* of an M-module A is a subgroup with the property that $mb \in B$ whenever $m \in M$ and $b \in B$.

2.4. Exercise. Suppose that $B \subseteq A$ is a sub-M-module, and $a \in A$. Show that the quotient group A/B is naturally an M-module. State and prove the First Isomorphism Theorem for M-modules.

Definition. Let X be a left M-set and let A an abelian group. We define the left M-module $A^{(X)}$ to be the abelian group of all finitely supported functions $\overline{a} : X \to A; x \mapsto \overline{a}_x$ with M action defined as follows:

$$(m\overline{a})_x = \sum_{m*y=x} \overline{a}_y.$$

If $\overline{a} \in A^{(X)}$, we can represent $\overline{a}$ as a formal sum

$$\overline{a} = \sum_{x \in X} \overline{a}_x x.$$

Here, of course, only finitely many of the $\overline{a}_x$ are non-zero. In this notation, the action of M can be expressed in the (possibly more transparent) form:

$$m \sum_{x \in X} \overline{a}_x x = \sum_{x \in X} \overline{a}_x mx.$$

2.5. Exercise. Let A be an abelian group. Define $\epsilon : A \to A^{(M)}$ by $\epsilon(a)_e := a$ and $\epsilon(a)_m := 0$ if $m \neq e$ (e being the neutral element of M). Show that $A^{(M)}$ has the following universal mapping property: if $\phi : A \to B$ is any group homomorphism from A to an M-module B, then there is a unique M-module morphism $\overline{\phi} : A^{(M)} \to B$ such that $\phi = \overline{\phi}\ \epsilon$. Show that $\left(\mathbb{Z}^{(I)}\right)^{(M)}$ is the free M-module on the set I.

M-Objects in general

The two constructions just described are special cases of a more general construction. Let $\mathbf{C}$ be a category. An M-object in $\mathbf{C}$ is an object C of $\mathbf{C}$ together with a monoid morphism from M to the monoid $\hom(C, C)$ of $\mathbf{C}$-morphisms from C to C. This is the same thing as a functor from M to $\mathbf{C}$. A morphism of M-objects is a $\mathbf{C}$ morphism between M-objects that commutes with the given actions of M on the domain and codomain. This is the same thing as a natural transform between functors.

Monoid Rings

Definition. Let R be a ring. The monoid ring $R[M]$ is the M-module $R^{(M)}$ equipped with the multiplication

$$(rr')_m := \sum_{k*\ell=m} r_k r'_\ell.$$

2.6. Exercises.

i) Show that this multiplication can also be expressed as follows:

$$\left(\sum_{m\in M} r_m m\right)\left(\sum_{n\in M} s_n n\right) = \sum_{m\in M}\sum_{n\in M} r_m s_n (m * n).$$

ii) Prove that $R[M]$ is a ring.

iii) Is $R[M]$ and M-object in the category of rings? Why or why not?

iv) Let A be a left M-module. Show that A is a left $\mathbb{Z}[M]$-module if we define:

$$\left(\sum_{m\in M} n_m m\right) a = \sum_{m\in M} n_m (ma).$$