Lecture 2. Algebraic constructions and categories (expanded version)

We define some categories and display some constructions that will play a role later.

M-sets

2.1. Definition. Let M be a monoid. A *left M-set* is a set X equipped with a function $M \times X \to X; (m, x) \mapsto m\,x$ such that $e\,x = x$ and $m\,(m'\,x) = (m * m')\,x$ for all $m, m' \in M$ and $x \in X$. A *left M-set morphism* is a function f between left M-sets that satisfies $m\,f(x) = f(m\,x)$.

Right M-sets and M-set morphisms are defined analogously. From now on, when we refer to an M-set without qualification (left or right), we mean a left M-set.

2.2. Examples.

a) Any monoid M is itself is a (left) M-set, where the action of $m \in M$ on $x \in M$ is defined by the operation of M: $m\,x = m * x$.
b) A congruence C on M is both a left and right M-set, with the actions defined as follows:
$$M \times C \to C; (m, (m', m'')) \mapsto (m * m', m * m'')$$
$$C \times M \to C; ((m', m''), m) \mapsto (m' * x, m'' * m).$$

2.3. Lemma. An equivalence relation E on M that is both a right and left M-set under the action described in 2.2.b) is a congruence.

Proof. Exercise. /////

2.4. Definition. A *sub-M-set* of an M-set X is a subset of X that is closed under the action of M. A sub-M-set of M is called an *ideal*.

2.5. Discussion. We will describe the "subobject classifier," Ω_M. Let X be an M-set, let $U \subseteq X$ be sub-M-set and let $x \in X$. We define the ideal $x\backslash U \subseteq M$ by

$$m \in x\backslash U \;\Leftrightarrow m\,x \in U.$$

Note that
$$(m_0\,x)\backslash U = m_0\backslash(x\backslash U) \quad \text{for all } m_0 \in M, \tag{1}$$
since
$$m \in (m_0\,x)\backslash U \;\Leftrightarrow m\,m_0\,x \in U \;\Leftrightarrow m\,m_0 \in x\backslash U \;\Leftrightarrow m \in m_0\backslash(x\backslash U).$$

Let Ω_M denote the set of all (left) ideals of M. The special case of (1) where $X = M$ and U is an ideal of M shows that Ω_M is an M-set, with the action defined by $(m, I) \mapsto m\backslash I$. The largest ideal of M—namely, M itself—will be denoted 1. The smallest ideal—namely, the empty set—will be denoted 0.

Let X be an M-set. We will show that there is a bijection between the set $\mathcal{P}_M(X)$ of sub-M-sets of X and the set $\hom_M(X, \Omega_M)$ of M-set morphisms from M to Ω_M. Let U

be a sub-M-set of X. Then the map $\chi_U : X \to \Omega_M$ defined by $\chi_U(x) := x\backslash U$ is an M-set morphism by (1), and

$$\chi_U^{-1}(1) = U. \tag{2}$$

Conversely, if $\phi : X \to \Omega_M$ is any M-set morphism, then

$$\begin{aligned} m \in x\backslash\phi^{-1}(1) &\Longleftrightarrow m\,x \in \phi^{-1}(1) \\ &\Longleftrightarrow \phi(m\,x) = 1 \\ &\Longleftrightarrow m\backslash\phi(x) = 1 \\ &\Longleftrightarrow 0 \in m\backslash\phi(x) \\ &\Longleftrightarrow m \in \phi(x) \end{aligned}$$

This shows that

$$\phi = \chi_{\phi^{-1}(1)}. \tag{3}$$

Now, (2) and (3) show that $U \mapsto \chi_U$ is a bijection with inverse $\phi \mapsto \phi^{-1}(1)$. /////

2.6. Definition. The *product of an I-indexed family of M-sets* $\{\, X_i \mid i \in I \,\}$ is the set-theoretic product, with the component-wise action: $(m\,y)_i := m\,(y_i)$, for $y \in \prod_{i\in I} X_i$. A *left M-set congruence on X* is an equivalence relation on X that is also sub-left-M-set of $X \times X$. The set of equivalence classes of a congruence C on S is naturally an M-set. It is denoted X/C and is called the *quotient of X by C*.

2.7. Facts. We describe *free M-sets*. Let E be a set. Then E is a left M-set under the trivial action $(m, y) \mapsto y$, $y \in E$. Suppose that X is a left M-set and $g : E \to X$ is any function. Then (prove this!) there is a unique left M-set morphism $\overline{g} : M \times E \to X$ such that $\overline{g}(m, y) = m\,g(y)$. Because of this universal mapping property, $M \times E$ is said to be the *free left M-set on E*.

2.8. Fact. Suppose that $\phi : M \to Q$ is a morphism of monoids. This induces a functor $\mathbf{M}_\phi$ from the category of left Q-sets to the category of left M-sets. If Y is a Q-set, $\mathbf{M}_\phi(Y)$ is simply Y endowed with the M-action $(m, y) \mapsto \phi(m)y$. To avoid confusion, we will use the notation $m \cdot_\phi y$ to denote this action. Thus, for a Q-set Y and $m \in M$,

$$m \cdot_\phi y := \phi(m)\,y.$$

We can define a functor $\mathbf{Q}_\phi$ from left M-sets to left Q-sets as follows. Let X be an M-set. The underlying set of X is $|X|$ and the free Q-set over $|X|$ is $Q \times |X|$. We set

$$\mathbf{Q}_\phi(X) := (Q \times |X|)/E(\phi, X),$$

where $E(\phi, X)$ is the smallest Q-set congruence on $Q \times |X|$ such that

$$(\phi(m), x) \sim_E (e, m'\,x) \text{ whenever } \phi(m) = \phi(m').$$

2.9. Theorem. $\mathbf{Q}_\phi$ is a functor and is left adjoint to $\mathbf{M}_\phi$.

Proof. To see this is a functor, suppose $f : X \to Y$ is an M-set morphism. Then there is a unique Q-set morphism

$$f_0 : Q \times |X| \to \mathbf{Q}_\phi(Y)\,;\; f_0(q,x) = \overline{(q, f(x))}.$$

Observe that if $\phi(m) = \phi(m')$, then

$$f_0(\phi(m), x) = \overline{(\phi(m), f(x))} = \overline{(e, m'\, f(x))} = \overline{(e, f(m'\, x))} = f_0(e, m'\, x).$$

This shows that $E(\phi, X)$ is contained in the kernel of f_0, and hence f_0 induces a Q-set morphism

$$\mathbf{Q}_\phi(f) : \mathbf{Q}_\phi(X) \to \mathbf{Q}_\phi(Y).$$

Now we will show that $\mathbf{Q}_\phi$ is left adjoint to $\mathbf{M}_\phi$, using the criterion of [S. MacLane, *Categories for the Working Mathematician*, Springer 1971. Chapter IV, Theorem 2 part(i), page 81]. Let X be an M-set. Define the morphism $\eta_X : X \to \mathbf{M}_\phi \mathbf{Q}_\phi(X)$ by $\eta_X(x) = \overline{(e, x)}$. This is an M-set morphism because

$$\eta_X(m\,x) = \overline{(e, m\,x)} = \overline{(\phi(m)\,e, x)} = m\,\overline{(e,x)} = m\,\eta_X(x).$$

Let Y be any Q-set and let $f : X \to \mathbf{M}_\phi(Y)$ be any M-set morphism. We will show that there is a unique Q-set morphism $\overline{f} : \mathbf{Q}_\phi(X) \to Y$ such that $f = \mathbf{M}_\phi(\overline{f}) \circ \eta_X$. There is a Q-set morphism $f_0 : Q \times |X| \to Y$ that is uniquely determined by setting $f_0\left((e,x)\right) = f(x)$. Now, $E(\phi, X) \subseteq \ker f_0$, because if $\phi(m) = \phi(m')$, then

$$f_0\left((\phi(m), x)\right) = \phi(m)\, f(x) = \phi(m')\, f(x) = m' \cdot_\phi f(x) = f(m'x) = f_0\left((e, m'x)\right).$$

Hence there is a unique Q-set morphism $\overline{f}$ such that $f_0 = \overline{f} \circ p$, where p is the canonical morphism $p : Q \times |X| \to \mathbf{Q}_\phi(x)$, and from the definition of f_0, it is clear that $f = \overline{f} \circ \eta_X$. /////

M-modules

Definition. Let M be a monoid. A *left M-module* is an abelian group A equipped with a function $M \times A \to A; (m,a) \mapsto ma$ such that $ea = a$ and $m(m'a) = (m * m')a$ and $m(a+b) = ma + mb$ for all $m, m' \in M$ and $a, b \in A$. Right M-module is defined analogously.

From now on, unless otherwise stated, "M-module" means left M-module. A function $\phi : A \to A'$ between M-modules is a *morphism of M-modules* if $\phi(m_1 a_1 + m_2 a_2) = m_1\phi(a_1) + m_2\phi(a_2)$ for all $m_1, m_2 \in M$ and $a_1, a_2 \in A$. A *sub-M-module* of an M-module A is a subgroup with the property that $mb \in B$ whenever $m \in M$ and $b \in B$.

Exercise. Suppose that $B \subseteq A$ is a sub-M-module, and $a \in A$. Show that the quotient group A/B is naturally an M-module. State and prove the First Isomorphism Theorem for M-modules.

Definition. Let X be a left M-set and let A an abelian group. We define the left M-module $A^{(X)}$ to be the abelian group of all finitely supported functions $\overline{a} : X \to A; x \mapsto \overline{a}_x$ with M action defined as follows:

$$(m\overline{a})_x = \sum_{m*y=x} \overline{a}_y.$$

If $\overline{a} \in A^{(X)}$, we can represent $\overline{a}$ as a formal sum

$$\overline{a} = \sum_{x \in X} \overline{a}_x x.$$

Here, of course, only finitely many of the $\overline{a}_x$ are non-zero. In this notation, the action of M can be expressed in the (possibly more transparent) form:

$$m \sum_{x \in X} \overline{a}_x x = \sum_{x \in X} \overline{a}_x m x.$$

Exercise. Let A be an abelian group. Define $\epsilon : A \to A^{(M)}$ by $\epsilon(a)_e := a$ and $\epsilon(a)_m := 0$ if $m \neq e$ (e being the neutral element of M). Show that $A^{(M)}$ has the following universal mapping property: if $\phi : A \to B$ is any group homomorphism from A to an M-module B, then there is a unique M-module morphism $\overline{\phi} : A^{(M)} \to B$ such that $\phi = \overline{\phi}\ \epsilon$. Show that $\left(\mathbb{Z}^{(I)}\right)^{(M)}$ is the free M-module on the set I.

M-Objects in general

The two constructions just described are special cases of a more general construction. Let $\mathbf{C}$ be a category. An M-object in $\mathbf{C}$ is an object C of $\mathbf{C}$ together with a monoid morphism from M to the monoid $\hom(C, C)$ of $\mathbf{C}$-morphisms from C to C. This is the same thing as a functor from M to $\mathbf{C}$. A morphism of M-objects is a $\mathbf{C}$ morphism between M-objects that commutes with the given actions of M on the domain and codomain. This is the same thing as a natural transform between functors.

Monoid Rings

Definition. Let R be a ring. The monoid ring $R[M]$ is the M-module $R^{(M)}$ equipped with the multiplication

$$(rr')_m := \sum_{k*\ell=m} r_k {r'}_\ell.$$

Exercises.

$i)$ Show that this multiplication can also be expressed as follows:

$$\left(\sum_{m\in M} r_m m\right)\left(\sum_{n\in M} s_n n\right) = \sum_{m\in M}\sum_{n\in M} r_m s_n (m * n).$$

ii) Prove that $R[M]$ is a ring.
iii) Is $R[M]$ and M-object in the category of rings? Why or why not?
iv) Let A be a left M-module. Show that A is a left $\mathbb{Z}[M]$-module if we define:

$$\left(\sum_{m\in M} n_m m\right) a = \sum_{m\in M} n_m (ma).$$