Lecture 3. Commutative monoids

Definition. A monoid M is *commutative* if $x * y = y * x$ for all $x, y \in M$.

From now on, monoids will be assumed commutative unless we explicitly say otherwise. We will use $+$ to denote the monoid operation and 0 to denote the neutral element. The n-fold sum $x + \cdots + x$ is denoted nx.

Definition.
1) M is *cancellative* if $x + y = x + z$ implies $y = z$ for all $x, y, x \in M$.
2) M is *torsionfree* if: for all $n = 1, 2, 3, \ldots$ and all $x, y \in M$, $nx = ny$ implies $x = y$. (Here, nx is an abbreviation for the n-fold sum: $x + x + \cdots + x$.)

The free group over a commutative monoid

2.1. Exercise. Let $\gamma_M : M \to G(M)$ be the free group over the commutative monoid M.

i) Let $\sim$ be the relation on $M \times M$ defined by

$$(a, b) \sim (c, d) \Leftrightarrow \text{for some } m \in M,\ a + d + m = b + c + m.$$

Show that $\sim$ is an equivalence relation and that it is closed under the action of $M \times M$:

$$(\ell, m)\,((a, b), (c, d)) := ((\ell + a, m + b), (\ell + c, m + d))\,.$$

Conclude that $\sim$ is a congruence on $M \times M$.

ii) Show that $(M \times M)/\sim$ is a group and that the morphism

$$\beta : M \to (M \times M)/\sim; a \mapsto [a, e]$$

has the universal mapping property of γ_M. (Hence $G(M) \cong (M \times M)/\sim$.)

ii) γ_M is injective if and only if M is cancellative. If M is cancellative *and* torsionfree, then $G(M)$ is torsionfree. However, there is a (non-cancellative) torsionfree M such that $G(M)$ is torsionfree.

Free commutative monoids.

$\mathbb{N}$ denotes the monoid of natural numbers: $\mathbb{N} = \{\,0, 1, 2, \ldots\}$. Let I be a set,. $\mathbb{N}^I$ denotes the product of I copies of $\mathbb{N}$, *i.e.*, the monoid of functions from I to $\mathbb{N}$. If $I = \{\,1, 2, \ldots, n\}$, we write $\mathbb{N}^n$. If $\alpha \in \mathbb{N}^I$ and $i \in I$ then $\alpha_i \in \mathbb{N}$ denotes the value of α at i.

If $i \in I$, then δ^i denotes the element of $\mathbb{N}^I$ that has value 1 at i and 0 elsewhere. Thus:

$$\delta^i_j = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{otherwise.} \end{cases}$$

$\mathbb{N}^{(I)}$ denotes the submonoid of $\mathbb{N}^I$ consisting of those functions that have finite support. Note that $\mathbb{N}^{(I)} = \langle\, \delta^i \mid i \in I\,\rangle$.

2.2. Fact. Let M be any commutative monoid and let $\phi : I \to M$ be any function. Then there is a unique monoid homomorphism $\overline{\phi} : \mathbb{N}^{(I)} \to M$ such that $\overline{\phi}(\delta^i) = \phi(i)$. (Thus, any finitely generated (commutative) monoid is isomorphic to $\mathbb{N}^n/C$ for some integer n and congruence C.)

Dickson's Lemma.

If $\alpha, \beta \in \mathbb{N}^I$, we write $\alpha \leq \beta$ if for each $i \in I$, $\alpha_i \leq \beta_i$. Thus,

$$\alpha \leq \beta \;\Leftrightarrow\; \exists \gamma \in \mathbb{N}^I \alpha + \gamma = \beta.$$

Note that any two elements α, β have a supremum $\alpha \vee \beta$ (the coordinatewise maximum of α and β) and infimum $\alpha \wedge \beta$ (the coordinate-wise minimum). Also, note that $\alpha \vee \beta$ and $\alpha \wedge \beta$ are in $\mathbb{N}^{(I)}$ if α and β are.

2.3. Lemma. *Every sequence $\alpha^1, \alpha^2, \cdots$ in $\mathbb{N}^n$ contains a weakly increasing subsequence $\alpha^{\sigma(1)} \leq \alpha^{\sigma(2)} \leq \cdots$. Thus, the set of $\leq$-minimal elements in any subset of $\mathbb{N}^n$ is finite.*

Proof. Let $\alpha^i = (\alpha^i_1, \ldots, \alpha^i_n)$, $i = 1, 2, \ldots$, be a sequence of elements of $\mathbb{N}^n$. This contains a subsequence $\alpha^{\sigma(i)}$ in which the last coordinate $\alpha_n^{\sigma(i)}$ is weakly increasing. This, in turn, contains a subsequence in which the $(n-1)$th coordinate is weakly increasing. After n steps, we have a subsequence satisfying the required condition. The last assertion follows because there are no order relations among the minimal elements of a set. If there were infinitely many, one could form a sequence from them that had no weakly increasing subsequence. /////

Congruences on commutative monoids: generators

2.4. Fact. Let M be a monoid and let B be a subset of $M \times M$. Let $X(B)$ be the set of all pairs (x, x') such that either $x = x'$ or there is a finite sequence:

$$x = y_1 + x_1 \sim y_1 + x'_1 = y_2 + x_2 \sim y_2 + x'_2 = \cdots = y_k + x_k \sim y_k + x'_k = x'$$

where for each $i = 1, \ldots, k$, $y_i \in M$ and either $(x_i, x'_i) \in B$ or $(x'_i, x_i) \in B$. Then $X(B) = \langle\langle B \rangle\rangle$, the congruence generated by B

Proof. In the statement of the fact, we used the symbol $\sim$ simply as a marker between elements in the sequence. But note that wherever it occurs, the elements on either side are equivalent under any congruence containing B. Hence, $\langle\langle B \rangle\rangle$ contains $X(B)$. Now, it is easy to see that $B \subseteq X(B)$ and that $X(B)$ is an equivalence relation that is closed under the M-action $(x, x') \mapsto (m + x, m + x')$, so $X(B)$ is a congruence. Hence, $X(B)$ contains $\langle\langle B \rangle\rangle$. /////

Ideals, and the congruences associated with them.

Definition. A subset $I \subseteq M$ of a monoid M is an *ideal* if $x + y \in I$ whenever $x \in I$ and $y \in M$.

Here are a few basic facts about ideals:

i) An intersection of ideals is an ideal. Thus, if $X \subseteq M$ there is a smallest ideal containing X. It is called the *ideal generated by* X and it is equal to $X + M := \{ x + m \mid x \in X, m \in M \}$.

ii) If $\phi : L \to M$ is a monoid morphism and $I \subseteq M$ is an ideal, then $\phi^{-1}(I)$ is an ideal in L. If $\phi : L \to M$ is surjective and $I \subseteq L$ is and ideal, then $\phi(I) \subseteq M$ is an ideal.

iii) A subset $I \subseteq \mathbb{N}^{(I)}$ is an ideal iff $z \geq x \in I \;\Rightarrow z \in I$. Hence, the ideal generated by $X \subseteq M$ is $\{y \in M \mid y \geq x, \text{ for some } x \in X\}$.

iv) By Dickson's Lemma, every ideal in $\mathbb{N}^n$ is finitely generated. Combined with *ii*), this shows that every ideal of a finitely generated monoid is finitely generated.

Suppose $I \subseteq M$ is an ideal. The equivalence relation on M that has I as its only non-singleton class is called the *Rees congruence of* I. The corresponding quotient monoid has an element ∞ with the property that $\infty + x = \infty$ for all $x \in M$. An element with this property is said to be *absorbing*.

The Rees congruences are the simplest of all congruences. Later, we will look at other other kinds.