Lecture 4.

This lecture is based on Chapter I of [Miller-Sturmfels]. We describe Stanley-Reisner ideals and consider some combinatorial data associated with them.

Recall:

- $I \subseteq \mathbb{N}^n$ is an ideal if $I + \mathbb{N}^n \subseteq I$.
- If $X \subseteq \mathbb{N}^n$, $\langle X \rangle := X + \mathbb{N} = \{\, \alpha \in \mathbb{N}^n \mid \xi \leq \alpha \text{ and } \xi \in X \,\}$.
- By Dickson's Lemma, every ideal in $\mathbb{N}^n$ has a unique set of minimal generators.

4.1. Fact. Suppose $I, J \subseteq \mathbb{N}^n$ are ideals. Then $I \cap J$ is generated by $\alpha \vee \beta$, where α is a minimal generator of I and β is a minimal generator of J. *Proof.* $\gamma \in I \cap J$ iff $\gamma \geq \alpha$ and $\gamma \geq \beta$ for some minimal generators $\alpha \in I$ and $\beta \in J$. /////

Stanley-Reisner ideals in $\mathbb{N}^n$

Associated to any subset $s \subseteq \{\, 1, 2, \ldots, n \,\}$, we have its characteristic function

$$\chi_s := \sum_{i \in s} \delta^i \in \mathbb{N}^n.$$

In general, we can view an arbitrary element of $\mathbb{N}^n$ as a subset of $\{\, 1, 2, \ldots, n \,\}$ with "multiplicities" attached to its members. An element all of whose multiplicities are either 0 or 1 is the characteristic function of a set. We call such an elemet *setlike*. For $\alpha \in \mathbb{N}^n$, we define $\operatorname{supp} \alpha := 1 \wedge \alpha$, this being the setlike element of $\mathbb{N}^n$ that vanishes at the same indices as α. Occasionally, we write $i \in \alpha$ to mean $\alpha_i \geq 1$.

4.2. Definition. A *simplicial complex* Δ is a collection of subsets of $\{\, 1, 2, \ldots, n \,\}$ such that $\tau \subseteq \sigma \in \Delta$ implies $\tau \in \Delta$. (See [MS], p.4, for related terminology.)

An ideal of $\mathbb{N}^n$ is "upwards closed", so the complement of an ideal is "downwards closed". (We say α is upwards from β if $\alpha = \beta + \gamma$ for some $\gamma \in \mathbb{N}^n$.) Thus, if $I \subseteq \mathbb{N}^n$ is an ideal, then the following collection of subsets of $\{\, 1, 2, \ldots, n \,\}$ is a simplicial complex:

$$\Delta_I := \{\, \sigma \in \{0,1\}^n \mid \sigma \notin I \,\}.$$

Note that if $I \subseteq J$, then $\Delta_J \subseteq \Delta_I$.

4.3. Definition. Suppose Δ simplicial complex. The *Stanley-Reisner ideal of* Δ is

$$I_\Delta := \langle\, \sigma \mid \sigma \notin \Delta \,\rangle = \{\, \alpha \in \mathbb{N}^n \mid \operatorname{supp} \alpha \notin \Delta \,\}.$$

4.3.1 Remark. $\alpha \notin I_\Delta$ iff $\operatorname{supp} \alpha \in \Delta$.

Remark. If $\Delta \subseteq \Delta'$, then $I_{\Delta'} \subseteq I_\Delta$. For all Δ, $\Delta = \Delta_{I_\Delta}$. For all I, $I \subseteq I_{\Delta_I}$.

4.4. Definition. I is said to be *prime* if its complement $\mathbb{N}^n \setminus I$ is closed under $+$.

4.5. Fact. Ideal $I \subseteq \mathbb{N}^n$ is prime if and only if its minimal elements are among the generators of $\mathbb{N}^n$. *Proof.* If a minimal element of I is not a generator of $\mathbb{N}^n$, it can be expressed as a sum of two elements that are strictly smaller, and hence are not in I./////

It follows that there is a bijection:

$$\begin{array}{ccc} \text{subsets of } \{\, 1, 2, \ldots, n \,\} & \longleftrightarrow & \text{prime ideals of } \mathbb{N}^n \\ \sigma & \longleftrightarrow & \langle\, \delta^i \mid i \in \sigma \,\rangle \end{array}.$$

Note that $\langle\, \delta^i \mid i \in \sigma \,\rangle = \{\, \alpha \in \mathbb{N}^n \mid \sigma \wedge \alpha \neq 0 \,\}$.

4.6. Lemma. *I_Δ is an intersection of prime ideals. Indeed,*

$$I_\Delta = \bigcap_{\sigma \in \Delta} \langle \delta^i \mid i \notin \sigma \rangle.$$

Proof. An element α of $\mathbb{N}^n$ belongs to the intersection $\Leftrightarrow$ for each $\sigma \in \Delta$, $\alpha \in \langle \delta^i \mid i \notin \sigma \rangle$ $\Leftrightarrow$ for each $\sigma \in \Delta$, there is $i \in \operatorname{supp} \alpha$ that is not in σ $\Leftrightarrow$ $\operatorname{supp} \alpha \notin \Delta \Leftrightarrow \alpha \in I_\Delta$. /////

Hilbert series

In this section, our point of view changes. We think in terms of the monomial ideal in $S := k[x_1, \ldots, x_n] = k[\mathbb{N}^n]$ rather than in terms of monoid ideals in $\mathbb{N}^n$. One can easily translate back and forth between these two kinds of object. In this subsection I_Δ will denote the ideal in S generated by the monomials x^σ, where $\sigma \in \mathbb{N}^n$ is setlike and $\sigma \notin \Delta$. An S-module M is $\mathbb{N}^n$-graded if $M = \bigoplus_{\beta \in \mathbb{N}^n} M_\beta$ and $x^\alpha M_\beta \subseteq M_{\alpha+\beta}$. If $\dim_k(M_\alpha) < \infty$ for all α, the $\mathbb{N}^n$-graded Hilbert series is:

$$H(M; x) := H(M; x_1, x_2, \ldots, x_n) := \sum_{\alpha \in \mathbb{N}^n} \dim_k(M_\alpha) x^\alpha.$$

Facts.

1. $H(S; x) = \prod_{i=1}^{n} (1 + x_i + x_i^2 + \cdots) = \prod_{i=1}^{n} \frac{1}{1 - x_i}$
2. $H(\langle x^\alpha \rangle; x) = x^\alpha H(S; x)$
3. Observe that $\sum \{ x^\alpha \mid \operatorname{supp} \alpha = \sigma \} = \prod_{i \in \sigma} \frac{x_i}{1 - x_i}$. Thus,

$$\begin{aligned} H(S/I_\Delta; x) &= \sum \{ x^\alpha \mid x^\alpha \notin I_\Delta \} \\ &= \sum \{ x^\alpha \mid \operatorname{supp} \alpha \in \Delta \} \quad \text{(see 4.3.1)} \\ &= \sum_{\sigma \in \Delta} \prod_{i \in \sigma} \frac{x_i}{1 - x_i} \\ &= H(S; x) \sum_{\sigma \in \Delta} \Big(\prod_{i \in \sigma} x_i \cdot \prod_{i \notin \sigma} (1 - x_i) \Big). \end{aligned}$$

The *K-polynomial* of M is $\mathcal{K}(M; x) := H(M; x)/H(S; x)$. Thus,

$$\mathcal{K}(S/I_\Delta; x) = \sum_{\sigma \in \Delta} \Big(\prod_{i \in \sigma} x_i \cdot \prod_{i \notin \sigma} (1 - x_i) \Big).$$