Lecture 5. Generalities—very general—on graded algebras and modules

5.1. Definition. Suppose k is a ring and A is a k-module. let X be a set. An X-grading of A is a decomposition if A into a sum of submodules indexed by X:

i) An *X-graded k-module* is a sum of k-modules:

$$A = \bigoplus_{\xi \in X} A_\xi \ .$$

The elements of A_ξ are said to be *homogeneous of degree* ξ; we write $\deg(a) = \xi$ if $a \in A_\xi$. Since 0 belongs to all the A_ξ, 0 does not have a unique degree.

ii) Now suppose Γ is a monoid. A *Γ-graded k-algebra* is a k-algebra A which is Γ-graded as a k-module and which in addition satisfies:

$$A_\gamma A_\lambda \subseteq A_{\gamma+\lambda} \quad \text{and} \quad 1_A \in A_0.$$

iii) Let A be a Γ-graded k-algebra. A *(A,Γ)-graded-module* (also referred to as a *graded A-module*, when the grading is obvious from context) is an A-module that is Γ-graded as a k-module, *i.e.*,

$$M = \bigoplus_{\gamma \in \Gamma} M_\gamma \ ,$$

and which in addition satisfies

$$A_\gamma M_\lambda \subseteq M_{\gamma+\lambda} \ .$$

Examples.

i) Let $S := k[x_1, \ldots, x_n]$. Then S is an $\mathbb{N}^n$-graded k-algebra. Note that S is also a graded S-module.

ii) The monoid algebra $k[\Gamma]$ is Γ-graded. ($k[\Gamma]_\gamma$ is the free k-module generated by X^γ.)

iii) Suppose A is a Γ-graded algebra and $\phi : \Gamma \to \Lambda$ is a monoid homomorphism. Then A can be given a Λ-grading by defining for $\lambda \in \Lambda$:

$$A_\lambda := \bigoplus \{\, A_\gamma \mid \phi(\gamma) = \lambda \,\} \ .$$

For example, $\mathbb{N}^n \to \mathbb{N}; \alpha \mapsto \alpha_1 + \cdots + \alpha_n$ gives the usual grading by total degree on $k[x_1, \ldots, x_n]$.

iv) Let L and M be graded A-modules. Then, $L \oplus M$ is A-graded if we define

$$(L \oplus M)_\gamma := L_\gamma \oplus M_\gamma \ .$$

v) Let A be a Γ-graded k-algebra and let B be a Λ-graded k-algebra. Then $A \otimes_k B$ is a k-algebra with multiplication determined by $(a \oplus b)(c \oplus d) = (ac \oplus bd)$ and the distributive law. $A \otimes_k B$ has a grading by $\Gamma \times \Lambda$:

$$(A \otimes_k B)_{(\gamma,\lambda)} := A_\gamma \otimes_k B_\lambda \ .$$

With respect to this grading, $A \otimes_k B$ is a $(\Gamma \times \Lambda)$-graded-k-algebra.

$v)$ Suppose that L is an (A, Γ)-graded-module and M is a (B, Λ)-graded-module. Then $A \otimes_k B$ acts on $L \otimes_k M$ by $(a \otimes b)(\ell \otimes m) = a\ell \otimes bm$. With respect to this action, $A \otimes_k B$ is a $(A \oplus_k B, \Gamma \times \Lambda)$-graded-module.

5.2. Definition. Let S be a Γ-graded A-module. We can shift the grading by $\gamma \in \Gamma$. It is customary to denote the result $S(-\gamma)$, where the graded pieces are defined by:

$$S(-\gamma)_\delta := \bigoplus \{ S_\lambda \mid \lambda + \gamma = \delta \} .$$

If Γ is cancellative, this gives $S(-\gamma)_\delta := S_{\delta - \gamma}$ if $\delta \geq \gamma$ and $S(-\gamma)_\delta := 0$ otherwise. If $s \in S$ and the degree of s in S is 0, then the degree of s in $S(-\gamma)$ is γ.

5.3. Definition. Let A be a Γ-graded k-algebra. A homomorphism $\phi : M \to N$ of A-modules is said to be *homogeneous of degree* δ if for all $\gamma \in \Gamma$:

$$\phi(M_\gamma) \subseteq N_\gamma + \delta .$$

The category of (A, Γ)-graded-modules and homogeneous homomorphisms of degree 0 is denoted $\mathcal{M}_{A,\Gamma}$ or just $\mathcal{M}_A$ if Γ is understood.

Let L be an (ungraded) A-submodule of an (A, Γ)-graded-module M. L is said to be a graded submodule of M if L is (A, Γ)-graded and the inclusion map is homogeneous, *i.e.*, $L_\gamma \subset M_\gamma$, *i.e.*, L is generated by homogeneous elements.

If $\phi : M \to N$ is a morphism of $\mathcal{M}_{A,\Gamma}$, then $\ker \phi$ is a graded submodule of M. If L is a graded submodule of M, then $M/L = \bigoplus_{\gamma \in \Gamma} M_\gamma / L_\gamma$ is an (A, Γ)-graded-module.

Free modules

The A-module $\bigoplus_n A$ has generators $\epsilon^i := (0, \ldots, 0, 1_A, 0, \ldots, 0)$. Given an A-module M (not graded) and a set of elements $m_i \in M$, there is a unique A-module map $\phi : \bigoplus_n A \to M$ such that $\phi(\epsilon^i) = m_i$.

Suppose, now, that A is Γ-graded. Then, so is $\bigoplus_n A$ (see Example iv). The degree of each ϵ_i is 0. Unless $\deg(m_i) = 0$ for all i, however, ϕ will not be a morphism of $\mathcal{M}_{A,\Gamma}$. To remedy this, we shift the grading on $\bigoplus_n A$, using $\bigoplus_{i=1}^n A(-\deg(m_i))$ instead.