Lecture 6. Graded morphisms

Part A. Monomial matrices

In the first part of this lecture, we describe the material presented in Miller-Sturmfels, pages 11–13. The goal is to develop a efficient format for specifying a homogeneous degree 0 morphism

$$\phi : \bigoplus_i A(-\gamma_i) \to \bigoplus_j A(-\delta_j),$$

where A is a Γ-graded k-algebra. Such morphisms occur in a free resolution of a (A, Γ)-graded-module. This explains our interest in this question.

To create the description, let ϵ_i (respectively, ϵ_j) be the generator of $A(-\gamma_i)$ (respectively, $A(-\delta_j)$). Its degree is γ_i (respectively, δ_j). Then $\phi(\epsilon_j) = \sum_j a_{ji}\epsilon_j$, where $\delta_j + \deg(a_{ji}) = \gamma_i$. Now,

$$\begin{aligned} \phi\left(\sum_i b_i\epsilon_i\right) &= \sum_i b_i \left(\sum_j a_{ji}\epsilon_j\right) \\ &= \sum_j \left(\sum_i a_{ji}b_i\right)\epsilon_j \ . \end{aligned}$$

If we let b denote the column vector $(b_i)^T$ and interpret it as an element of $\bigoplus_i A(-\gamma_i)$ by identifying the column with 1 in the i^{th} place and 0 elsewhere with ϵ_i, and if furthermore we view $a = (a_{ji})$ as a matrix with entries from A, then

$$\phi(b) = ab \ .$$

Additional compression of notation is possible under additional assumptions on A. We write $\delta_j \preceq \gamma_i$ to mean that $\delta_j + \xi = \gamma_i$ has a solution. If the solution is unique, we denote it $\xi = \gamma_i - \delta_j$. Assuming a unique solution, $a_{ij} \in A_{\gamma_i - \delta_j}$. Now, suppose k is a field, A_γ is one-dimensional for all $\gamma \in \Gamma$ (as is the case when A is a monoid algebra, $A = k[\Gamma]$), and suppose further that Γ is cancellative. Then the degrees γ_i and δ_j determine the degree of a_{ji} uniquely: $\deg(a_{ji}) = \gamma_i - \delta_j$, provided $\delta_j \preceq \gamma_i$. Once a basis element for each A_γ is chosen, the the coefficient a_{ji} will be determined completely by an element $\lambda_{ji} \in k$. Thus, the mapping ϕ can be represented by a matrix with entries in k. For example, if $A = k[\Gamma]$, then:

$$a_{ji} = \begin{cases} \lambda_{ji}X^{\gamma_i - \delta_j} & \text{if } \delta_j \preceq \gamma_i; \\ 0 & \text{otherwise.} \end{cases}$$

The grade vectors (γ_i) and (δ_j) and the matrix (λ_{ji}) determine ϕ completely.

See the example on page 13 of [Miller-Sturmfels], which illustrates the presentation of several morphisms in the manner just described. (The text asserts that morphisms on page 13 constitute a minimal free resolution. The reader should verify this.)

Part B. The Koszul complex

We describe the Koszul resolution of k as a $k[\mathbb{N}^n]$-module. Throughout this example, k is a field.

As background, we review the definition of the reduced (or augmented) chain complex of an oriented simplicial complex Δ. Let $F_i = F_i(\Delta)$ denote the set of i-dimensional faces of Δ, *i.e.*, elements of Δ of cardinality $i+1$. For $\sigma \in F_i$, let $e_\sigma \in k^{F_i}$ be the function that has value 1 at σ and is 0 otherwise. Define $\partial_i : k^{F_i} \to k^{F_{i-1}}$ by:

$$\partial_i(e_\sigma) = e_{\sigma\backslash 1} - e_{\sigma\backslash 2} + e_{\sigma\backslash 3} - \cdots + (-1)^i e_{\sigma\backslash i+1} , \tag{1}$$

where $\sigma\backslash j$ denotes σ with its j^{th} element (in the order determined by the given orientation) dropped. Then, $\widetilde{\mathcal{C}}_\bullet(\Delta; k)$ is the complex:

$$0 \longleftarrow k^{F_{-1}} \xleftarrow{\partial_0} k^{F_0} \xleftarrow{\partial_1} k^{F_1} \xleftarrow{\partial_2} \cdots \xleftarrow{\partial_d} k^{F_d} \longleftarrow 0 ,$$

where d is the largest dimension of any face of Δ. The reduced homology of Δ is the sequence of vector spaces:

$$\widetilde{H}_i(\Delta, k) = \ker(\partial_i)/\operatorname{im}(\partial_i) .$$

Assume the vertices of Δ are labeled $1, 2, \ldots, n$. We modify the above construction by using $S := k[\mathbb{N}^n]$ in place of k and assigning grades in an appropriate way. Explicitly, let

$$\mathbb{K}_i(\Delta) := \bigoplus_{\sigma\in F_{i-1}} S(-\sigma).$$

Here, we are using σ to refer to an element of $\mathbb{N}^n$. For example, $(1, 0, \ldots, 0) \in F_0$ is the first vertex in Δ, and $(1, 1, 0, 0, \ldots, 0)$ is a one-dimensional simplex, which might or might not be in F_1. Note that we have numbered the free S-modules $\mathbb{K}_i(\Delta)$ in such a way that i refers to the number of elements in each of the faces involved in defining $\mathbb{K}_i(\Delta)$. In particular, $\mathbb{K}_0(\Delta) = S$ is the free module with one generator in degree $(0, 0, \ldots, 0)$, corresponding to the unique empty face $(0, 0, \ldots, 0)$. Also, $\mathbb{K}_1(\Delta) = S^n$. Thus, the numbering of $\mathbb{K}_\bullet(\Delta)$ is shifted from $\widetilde{\mathcal{C}}_\bullet$.

For $\sigma \in F_{i-1}$, we let ϵ_σ denote the generator of $S(-\sigma) \subseteq \mathbb{K}_i(\Delta)$. Finally, we define $\partial_i : \mathbb{K}_i(\Delta) \to \mathbb{K}_{i-1}(\Delta)$ by:

$$\partial_i(\epsilon_\sigma) = X^{\sigma[1]}\epsilon_{\sigma\backslash 1} - X^{\sigma[2]}\epsilon_{\sigma\backslash 2} + X^{\sigma[3]}\epsilon_{\sigma\backslash 3} - \cdots + (-1)^{i-1}X^{\sigma[i]}\epsilon_{\sigma\backslash i} , \tag{2}$$

where $\sigma[j] := \sigma - (\sigma \setminus j)$. (The choice of coefficient is precisely what is needed to balance degrees.) Then the complex $\mathbb{K}_\bullet(\Delta)$ is the following:

$$0 \longleftarrow \mathbb{K}_0(\Delta) \xleftarrow{\partial_1} \mathbb{K}_1(\Delta) \xleftarrow{\partial_2} \mathbb{K}_2(\Delta) \xleftarrow{\partial_3} \cdots \xleftarrow{\partial_{d+1}} \mathbb{K}_{d+1}(\Delta) \longleftarrow 0 . \tag{2.1}$$

Let $\mathbb{K}_{\bullet\,\alpha}(\Delta)$ denote the part of $\mathbb{K}_\bullet(\Delta)$ in degree $\alpha \in \mathbb{N}^n$. Thus,

$$\mathbb{K}_{i\,\alpha}(\Delta) \subseteq \bigoplus\{\, S(-\sigma)_\alpha \mid \sigma \in \Delta\,,\ \operatorname{card}\sigma = i \;\&\; \sigma \le \alpha \,\} ,$$

and $\mathbb{K}_{i\,\alpha}(\Delta)$ has basis, $\{\, e_{\sigma\,\alpha} := X^{\alpha-\sigma}\epsilon_\sigma \in S(-\sigma)_\alpha \mid \sigma \in \Delta\,,\ \operatorname{card}\sigma = i \;\&\; \sigma \le \alpha \,\}$. Let us write $\partial_{i\alpha}$ for the restriction of ∂_i to $\mathbb{K}_{i\,\alpha}(\Delta)$. Then, $\operatorname{im}(\partial_{i\alpha}) \subseteq \mathbb{K}_{(i-1)\,\alpha}(\Delta)$. Multiplying (2) by $X^{\alpha-\sigma}$, we see

$$\partial_{i\alpha}(e_\sigma) = e_{\sigma\setminus 1} - e_{\sigma\setminus 2} + e_{\sigma\setminus 3} - \cdots + (-1)^{i-1} e_{\sigma\setminus i}\,. \tag{3}$$

Thus, $\mathbb{K}_{\bullet\,\alpha}(\Delta)$ is the same as the complex $\widehat{\mathcal{C}}(\Delta \wedge \alpha, k)$ (except it is shifted—$\mathbb{K}_{\bullet\,\alpha}(\Delta)$ is indexed starting from 0 rather that -1), where $\Delta \wedge \alpha := \{\, \sigma \wedge \alpha \mid \sigma \in \Delta \,\} = \{\, \sigma \in \Delta \mid \sigma \subseteq \operatorname{supp}\alpha \,\}$.

Proposition. *Let Δ be the $(n-1)$-simplex, i.e. the set of all subsets of $\{\,1, 2, \ldots, n\,\}$. Then the complex $\mathbb{K}_\bullet := \mathbb{K}_\bullet(\Delta)$ is a minimal free resolution of $S/(x_1, \ldots, x_n)$.*

Proof. It is necessary only to show that (2.1) is exact. Since Δ is the $(n-1)$-simplex, $\Delta \wedge \alpha$ is itself a simplex and therefore has 0 homology. Thus, $\mathbb{K}_{\bullet\,\alpha}$ is exact for each α, and this establishes the result we need. /////

Question for listeners/readers. Fill in the missing portion of the proof concerning minimality. Refer to Definition 1.24 on page 12 of Miller-Sturmfels.