Lecture 7. Betti Numbers

In this talk, $S = k[\mathbb{N}^n]$. All modules will be $(S, \mathbb{N}^n)$-graded-modules. Let

$$0 \longleftarrow M \longleftarrow F_0 \xleftarrow{\phi_1} F_1 \xleftarrow{\phi_2} \cdots \xleftarrow{\phi_\ell} F_\ell \longleftarrow 0$$

be an augmented free resolution of M in which the matrices a_{ji} defining the maps ϕ have no nonzero entries in degree 0. In other words, we are assuming that the resolution is *minimal*; see [Miller-Sturmfels]=[MS], page 12. (The minimal free resolution of M is unique up to isomorphism; see the references in [MS] or [Bruns-Herzog], page 36.) Now suppose

$$F_i = \bigoplus_{\alpha \in \mathbb{N}^n} S(-\alpha)^{\beta_{i\alpha}},$$

with $\beta_{i\alpha} \in \mathbb{N}$ being the number of times that $S(-\alpha)$ occurs as a summand of F_i. These numbers, which are completely determined by M, are called the *graded Betti numbers of* M. (Graded Betti numbers may be defined in a similar manner for graded rings other than S, but there are restrictions on the ring that must be met, without which uniqueness may fail; see [Bruns-Herzog], page 37.) The goal of this talk is to demonstrate how the Tor functor can be used to compute the $\beta_{i\alpha}$.

We suggest that the reader review Lecture 5 for information on tensor products $M \otimes_A N$. (Some parts have been revised since the first version was posted, and comments intended to clarify the following discussion have been included.)

Let M and N be A-modules. Recall that $\operatorname{Tor}^A(M, N)$ is the homology of $F_\bullet \otimes_A N$, where $F_\bullet$ is a free resolution of M:

$$\operatorname{Tor}_i^A(M, N) = \ker(\phi_i \otimes 1) / \operatorname{im}(\phi_{i+1} \otimes 1).$$

When the free resolution is graded, then so is $\operatorname{Tor}_i^A(M, N)$, as the reader should check.

Lemma. *Suppose M is a $(S, \mathbb{N}^n)$-graded k-algebra. Then $\beta_{i\alpha} = \dim_k \operatorname{Tor}_i^S(M, k)_\alpha$.*

Proof. Note that k itself is an $(S, \mathbb{N}^n)$-graded-module, concentrated in degree 0. Thus, $S(-\alpha) \otimes_S k = k(-\alpha)$. Also, note that if $\phi : S(-\alpha) \to S(-\gamma)$ is morphism of degree 0, then $\phi \otimes 1_k : S(-\alpha) \otimes k \to S(-\gamma) \otimes k$ is the zero map, unless $\alpha = \gamma$. Thus, if $F_\bullet$ is a minimal resolution, then

$$\phi_i \otimes 1_k = 0$$

for $i = 0, 1, \ldots$, since $F_\bullet$ being minimal, every a_{ji} has degree > 0. Accordingly,

$$\operatorname{Tor}_i^A(M, k) = \bigoplus_{\alpha \in \mathbb{N}^n} k(-\alpha)^{\beta_{i\alpha}},$$

and in degree α, we have just $k(-\alpha)^{\beta_{i\alpha}}$. This proves the claim. /////

Recall that $\operatorname{Tor}^A(M,N) \cong \operatorname{Tor}^A(N,M)$. This enables us to compute $\operatorname{Tor}^A(M,N)$ from a resolution of N rather than M. In particular, we can compute $\operatorname{Tor}^A(M,k)$ by using the Koszul resolution $\mathbb{K}_\bullet$ of k to compute $\operatorname{Tor}^A(k,M)$. This works out particularly nicely when M is a monomial ideal of S. In fact, the Betti numbers of I turn out to be the Betti numbers of certain simplicial complexes associated with I.

Definition. Let $I \subseteq k[\mathbb{N}^n]$ be a monomial ideal. (We can also view I simply as a monoid ideal of $\mathbb{N}^n$.) Let $\gamma \in \mathbb{N}^n$. Then

$$\mathbf{K}^\alpha(I) := \left\{ \tau \in \{0,1\}^n \mid \alpha - \tau \in I \right\}.$$

Theorem. ([MS], 1.34). $\beta_{i\alpha}(I) = \dim_k \widetilde{H}_{i-1}(\mathbf{K}^\alpha(I), k)$.

Proof. Recall that the Koszul resolution $\mathbb{K}_\bullet$ is:

$$0 \xleftarrow{\phi_0} S \xleftarrow{\phi_1} \bigoplus_{|\tau|=1} S(-\tau) \xleftarrow{\phi_2} \bigoplus_{|\tau|=2} S(-\tau) \xleftarrow{\phi_3} \cdots.$$

Here, we are assuming that $\tau \in \{0,1\}^n$. Observe that $\operatorname{im}\phi_1 = \langle x_1, \dots, x_n \rangle$. Now,

$$\begin{aligned}\beta_{i\alpha}(I) &= \dim_k \operatorname{Tor}_i^S(k,I)_\alpha \\ &= \dim_k(\text{degree-}\alpha \text{ part of the } i^{\text{th}} \text{ homology of } \mathbb{K}_\bullet \otimes_S I).\end{aligned}$$

Because $I \subseteq S$ and $\mathbb{K}_\bullet$ is a sum of free modules, $\mathbb{K}_\bullet \otimes_S I \subseteq \mathbb{K}_\bullet \otimes_S S = \mathbb{K}_\bullet$, so $(\mathbb{K}_\bullet \otimes_S I)_\alpha \subseteq (\mathbb{K}_\bullet)_\alpha$. We saw in Lecture 6 that $(\mathbb{K}_\bullet)_\alpha$ is the reduced chain complex of $\operatorname{supp}\alpha$ (shifted up in homological degree). We shall show that $(\mathbb{K}_\bullet \otimes_S I)_\alpha$ is the reduced chain complex of $\mathbf{K}^\alpha(I) \subseteq \operatorname{supp}\alpha$. Indeed,

$$\begin{aligned}(\mathbb{K}_i \otimes_S I)_\alpha &= \left(\bigoplus_{|\tau|=i} S(-\tau) \otimes I \right)_\alpha \\ &= \bigoplus_{|\tau|=i} I(-\tau)_\alpha \\ &\cong k^{\mathbf{K}_i^\alpha(I)},\end{aligned}$$

since

$$I(-\tau)_\alpha = \begin{cases} k, & \text{if } \alpha - \tau \in I; \\ 0, & \text{otherwise.} \end{cases}$$

Thus, $(\mathbb{K}_\bullet \otimes_S I)_\alpha$ is the reduced chain complex of $\mathbf{K}^\alpha(I)$, re-numbered so as to start in degree 0 rather than -1. /////