Equational classes of f-rings with unit: disconnected classes.

James J. Madden
LSU, Baton Rouge

Dedication: To the memory of Paul Conrad.

Abstract. This paper introduces several families of equational classes of unital f-rings that are defined by equations that impose conditions on the elements between 0 and 1. We investigate the portion of the lattice of equational classes of f-rings that involves these classes.

1. Introduction. All rings in this paper are commutative and have a unit. The real numbers, the rational numbers and the integers are denoted $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{Z}$, respectively. If A is a ring with auxiliary operation $\vee$ (*e.g.*, an f-ring), then $\mathbf{HSP}(A)$ denotes the equational class generated by A, *i.e.*, the class of all $\vee$-rings that satisfy all the equational identities that are true in A. By Birkhoff's Theorem, $\mathbf{HSP}(A)$ coincides with the class of all $\vee$-rings that are isomorphic to a homomorphic image of a sub-$\vee$-ring of a product of copies of A, whence the notation.

We call a totally-ordered ring *singular* if it has no elements strictly between 0 and 1, and we call it *disconnected* if every element between 0 and 1 is infinitely close to 0 or to 1. We call an f-ring singular (respectively, disconnected) if it is a sub-f-ring of a product of singular (respectively, disconnected) totally-ordered rings.

Let $B := \mathbb{Z}[X]$, ordered so that X is infinitely small, and let $B_n := B/(X^n)$. In this paper, we show the following:

i) The singular f-rings form an equational class, as do the disconnected f-rings. These classes will be denoted $\mathbf{SFR}$ and $\mathbf{DFR}$, respectively.

ii) If A is an f-ring, then $\mathbb{Q} \in \mathbf{HSP}(A)$ if and only if A is *not* disconnected.

iii) Let $\mathbf{D}_n\mathbf{FR}$ denote the class of f-rings defined by the condition that infinitesimals have nilpotency degree n, where $n = 2, 3, \ldots$. Each such class is equational, and all these classes lie between $\mathbf{SFR} = \mathbf{D}_1\mathbf{FR}$ and $\mathbf{DFR}$. (Obviously, $B_n \in \mathbf{D}_n\mathbf{FR}$, but $B_n \notin \mathbf{D}_{n-1}\mathbf{FR}$).

iv) For $n > 3$, the classes $\mathbf{D}_n\mathbf{FR}$ are incomparable with $\mathbf{HSP}(\mathbb{Q})$. ($\mathbf{HSP}(\mathbb{Q})$is the only proper equational class of f-rings that has previously received any significant attention).

v) $\mathbf{D}_1\mathbf{FR}$ and $\mathbf{D}_2\mathbf{FR}$ are contained in $\mathbf{HSP}(\mathbb{Q})$.

vi) Between $\mathbf{HSP}(\mathbb{Z})$ and $\mathbf{SFR}$, there are infinitely many pairwise incomparable equational classes. Between $\mathbf{HSP}(B_n)$ and $\mathbf{D}_n\mathbf{FR}$, there are infinitely many pairwise incomparable equational classes none of which are contained in $\mathbf{D}_{n-1}\mathbf{FR}$. Similarly, between $\mathbf{HSP}(B)$ and $\mathbf{DFR}$, there are infinitely many pairwise incomparable equational classes, none of which are contained in any $\mathbf{D}_n\mathbf{FR}$.

The following diagram illustrates some of the containments described above. An upward-pointing arrow means a containment of the lower class in the one above. We do not intend the diagram to be interpreted as saying anything about the joins or meets of equational classes.

$$\begin{tikzcd}
 & \mathbf{DFR} & & \\
\mathbf{HSP}(B) \arrow[ur] & & \mathbf{D}_n\mathbf{FR} \arrow[ul] & \\
 & \mathbf{HSP}(B_n) \arrow[ul] \arrow[ur] & & \mathbf{SFR} \arrow[ul] \\
 & & \mathbf{HSP}(\mathbb{Z}) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

In the remainder of the introduction, we fix more notation and provide some background information. A *positive cone* in a ring A is a subset $P \subseteq A$ such that

$$P + P \subseteq P\,, \quad PP \subseteq P\,, \quad A^2 \subseteq P \quad \& \quad P \cap -P = \{0\}.$$

It is well-known that a positive cone P induces a a ring order on A in which $0 \le a$ if and only if $a \in P$, and that every ring order comes from a positive cone. A totally-ordered ring—or *toring* for short—is a ring equipped with a total order, *i.e.*, a positive cone P such that $-P \cap P = \{0\}$. A ring is *reduced* if it has no nilpotents. A reduced toring is a domain, since if $0 < a \le b$ and $0 = a\,b$, then $a^2 = 0$. By the same token, the nilradical N of a toring A is convex and the residue ring A/N is a totally-ordered domain.

By the language of $\vee$-*rings*, we mean the first-order language with constants 0 and 1, a unary function symbol, $-$, and binary function symbols $+$, $\cdot$ and $\vee$. An f-ring is a member of the equational class of $\vee$-rings whose laws are the laws of commutative rings together with the laws below concerning $\vee$. (The notation used in the last law is explained after the laws are stated.)

s_1) $X \vee (Y \vee Z) = (X \vee Y) \vee Z$,

s_2) $X \vee Y = Y \vee X$,

s_3) $X \vee X = X$,

a) $(X \vee Y) + Z = (X + Z) \vee (Y + Z)$,

m) $X^+ Y^+ \wedge Y^- = 0$.

The first four of these laws say that the additive group of an f-ring is an ℓ-group (*i.e.*, a lattice-ordered group). In any ℓ-group, the operation $\wedge$ is defined by: $x \wedge y := -(-x \vee -y)$. Also, we define $x^+ := x \vee 0$, $x^- := (-x) \vee 0$ and $|x| := x^+ \vee x^-$. Under $\vee$ and $\wedge$, any ℓ-group is a distributive lattice. Note that m) implies that if $0 \le x$ and $0 \le y$, then $0 \le x\,y$, but it is actually a much stronger condition.

Any totally-ordered ring is an f-ring with respect to the operation $x \vee y := \max\{x, y\}$. In general an f-ring need not be totally-ordered, but it is a theorem (first proved by Birkhoff and Pierce, [BP]) that any f-ring is a sub-f-ring of a product of totally ordered rings. Thus, any lattice-ring identity that is violated in an f-ring is violated in a toring. This provides a way to treat questions about f-rings by rephrasing them as questions about torings (and vice versa). For an overview of f-rings, see [BKW], Chapter 9.

2. Review of earlier results. In this section, we summarize what was known about equational classes of f-rings (with unit) prior to the contributions of the present paper. (We shall not number propositions that are quoted from other sources.) Henriksen and Isbell [HI] showed that any element w of the free f-ring on $\{X_1, \ldots, , X_n\}$ can be expressed in the form:

$$w = \bigvee_{i=1}^{k} \bigwedge_{j=1}^{\ell_i} f_{ij} \ , \quad \text{where} \ \ f_{ij} \in \mathbb{Z}[X_1, \ldots, X_n].$$

Accordingly, every f-ring identity is of the form $w = 0$, where w is as above. Let us write $w \leq 0$ as an abreviation for $w \vee 0 = 0$. Then, $w = 0$ is equivalent to $w \leq 0 \ \& - w \leq 0$. Noting that both w and $-w$ may be written as suprema of infima and that $\bigvee_{i=1}^{k} w_i \leq 0$ is equivalent to the conjunction of the equations $w_i \leq 0$, we conclude:

Proposition. [HI]. *Every f-ring identity is equivalent to a conjunction of equations of the form*

$$f_1 \wedge \ldots \wedge f_s \leq 0 \ , \quad f_i \in \mathbb{Z}[X_1, \ldots, X_n].$$

The class of all f-rings is denoted **FR**. Since every non-trivial f-ring contains the integers, $\mathbb{Z}$, the smallest non-trivial equational class of f-rings is $\mathbf{HSP}(\mathbb{Z})$, the class of f-rings satisfying all $\vee$-ring identities true in $\mathbb{Z}$. For example, each unsolvable Diophantine equation produces a law of $\mathbb{Z}$, for if $f \in \mathbb{Z}[X_1, \ldots, X_n]$ has no integer solutions, then for all $x_1, \ldots, x_n \in \mathbb{Z}$,

$$1 - |f(x_1, \ldots, x_n)| \leq 0.$$

The existence of these laws was pointed out in [HI], but up to the present the literature contains nothing further about them.

Among the laws satisfied by $\mathbb{Z}$ is the following:

$$X^2 \wedge X = X. \tag{S}$$

This says that if $0 \leq x \leq 1$, then $x = 0$ or $x = 1$. The class of *singular* f-rings, denoted **SFR** is the class defined by the single equation (S). A toring is in this class if and only if 1 is the smallest positive element. Any f-ring in **SFR** is reduced, and obviously $\mathbf{HSP}(\mathbb{Z})$ is contained in **SFR**.

Henriksen and Isbell [HI] showed that all totally-ordered fields satisfy the same $\vee$-ring identities, and that not all of these identities are implied by the f-ring identities. They called an f-ring that satisfies all such identities "formally real". From general facts of universal algebra, it follows that the class of formally real f-rings is $\mathbf{HSP}(\mathbb{Q})$.

No other work on the structure of the lattice of equational classes of f-rings with unit has been done, though there are a few results concerning the non-unital case; see [BKW], Chapter 9. The presently known equational classes in the unital case and the relations between them, then, are summarized in the following diagram:

$$\mathbf{FR} \supset \mathbf{HSP}(\mathbb{Q}) \supset \mathbf{SFR} \supseteq \mathbf{HSP}(\mathbb{Z}).$$

While we know little of the general structure of the lattice of equational classes of f-rings, quite a lot is known about the specific class $\mathbf{HSP}(\mathbb{Q})$.

- Any reduced f-ring is in $\mathbf{HSP}(\mathbb{Q})$, [HI]. This follows from the fact that any reduced f-ring is contained in a product of totally-ordered domains, and that every totally-ordered domain is contained in a totally-ordered field.
- A nice description of free formally real f-rings comes from general theorems in universal algebra, [HI]. The free formally real f-ring on n generators is isomorphic to the sub-f-ring of the f-ring of all functions from $\mathbb{Q}^n$ to $\mathbb{Q}$ that is generated by the projections. Thus, it is an f-ring of piecewise polynomial functions.
- **Proposition.** [HI]. *A toring A is formally real if and only if for any ring homomorphism $\phi : \mathbb{Z}[\, X_1, \ldots, X_n \,] \to A$, there is a total cone $T \subseteq \mathbb{Z}[\, X_1, \ldots, X_n \,]$ whose image under ϕ is contained in the positive cone of A.* ▪
- Isbell [I] showed that $\mathbf{HSP}(\mathbb{Q})$ does not have an equational base with finitely many variables. Indeed, he showed that for each integer $k \geq 3$, there are laws in $3k$ variables that are not implied by the laws in fewer variables.
- **Proposition.** [M]. *Let $\mathcal{E}$ be the set of f-ring identities of the form $\bigwedge F \leq 0$, where F is a finite subset of $\mathbb{Z}[X_1, X_2 \ldots]$ that is not contained in any positive cone. Then $\mathcal{E}$ is an equational base for the formally real f-rings.* ▪

3. New results. We now demonstrate the claims in the introduction, working through them in roughly the order that they were stated.

If A is a toring and $\alpha \in A$, then by $\mathbb{Z}[\alpha]$ we mean the smallest sub-toring of A containing α, *i.e.*, the elements of the form $f(\alpha)$, where f is a polynomial with coefficients in $\mathbb{Z}$.

Proposition 1. *If $\alpha \in \mathbb{R}$ and $0 < \alpha < 1$, then $\mathbb{Q} \in \mathbf{HSP}(\mathbb{Z}[\alpha])$.*

Proof. Let $C \subseteq \mathbb{Z}[\alpha]^{\mathbb{N}}$ be the ring of all convergent sequences. Given any $r \in \mathbb{R}$, $r > 0$, let $c_i(r) := \max\{\, n\alpha^i \mid n \in \mathbb{N} \;\&\; n\alpha^i \leq r \,\}$. Then, $c_i(r) \to r$ as $i \to \infty$. Thus, if $I \subset C$ is the ideal of sequences that converge to 0, we have $C/I \cong \mathbb{R}$. ▪

Proposition 2. *If A is a toring that contains an element α such that $1 < n\alpha < n-1$ for some $n \in \mathbb{N}$, then $\mathbb{Q} \in \mathbf{HSP}(A)$.*

Proof. Let $I \subset A$ be the ideal of infinitesimals, *i.e.*, $I = \{\, b \in A \mid \forall n \in \mathbb{N} \;\; -1 < nb < 1 \,\}$. By Hölder's theorem, the smallest convex subring of A/I containing 1 is isomorphic to a subring of $\mathbb{R}$. If $\overline{\alpha}$ is the residue of α in A/I, then $0 < \overline{\alpha} < 1$, so the result follows from Proposition 1. ▪

If $\mathbb{Q}$ is not in $\mathbf{HSP}(A)$, then the elements of A between 0 and 1 must be infinitely close to 0 or to 1. Remarkably, there is an equational law that imposes exactly this condition:

$$(3X - 1) \wedge (2 - 3X) \leq 0. \qquad (D)$$

We call an f-ring that satisfies (D) *disconnected*. The equational class of disconnected f-rings is denoted **DFR**. Figuratively, (D) says that the interval $(1/3, 2/3)$ in A is empty. This is impossible if A contains an element between 0 and 1 that is not infinitely close to 0 or to 1. Thus, it is clear that each law of the form

$$(qX - p) \wedge (p' - qX) \leq 0, \text{with } p, p', q \in \mathbb{Z} \text{ and } 0 < p < p' < q$$

is equivalent to (D).

Proposition 3. *If A is a toring, then $\mathbb{Q} \in \mathbf{HSP}(A)$ if and only if A violates (D).*

Proof. If A satisfies (D), then $\mathbb{Q}$ is not in $\mathbf{HSP}(A)$ since clearly $\mathbb{Q}$ violates (D) with $X = 1/2$. If $\alpha \in A$ is a counterexample to (D), then $1 < 3\alpha < 2$, so α satisfies the condition in the previous proposition. ∎

Nil-disconnected classes. Stronger than (D) is the condition that any element between 0 and 1 must be nilpotent of degree n—or have a difference from 1 that is nilpotent of degree n. This condition is equivalent to the following equational law:

$$(1 - 2X) \wedge X \wedge X^n \leq 0 \tag{D_n}$$

We call the equational class defined by (D_n) *nil-disconnected of exponent* n, and denote it $\mathbf{D}_n\mathbf{FR}$.

Note that (D_1), which says $(1 - 2X) \wedge X \leq 0$, is equivalent to the singular law (S). That is, a toring satisfies (D_1) if and only if it's singular. Clearly (D_n) implies (D_ℓ) for all $\ell > n$. Also, each (D_n) implies (D). Thus, we have a chain of equational classes:

$$\mathbf{DFR} \supseteq \cdots \supseteq \mathbf{D}_k\mathbf{FR} \supseteq \mathbf{D}_{n-1}\mathbf{FR} \supseteq \cdots \supseteq \mathbf{D}_1\mathbf{FR} = \mathbf{SFR}. \tag{1}$$

Recall the definition of B_n from the introduction. Evidently, B_n satisfies (D_n) but not (D_{n-1}). This shows that the containments in (1) are all proper. Also, we have the following chain of equational classes, where each class is contained in the corresponding class of the previously mentioned chain:

$$\mathbf{HSP}(B) \supset \cdots \supset \mathbf{HSP}(B_n) \supset \mathbf{HSP}(B_{n-1}) \supset \cdots \supset \mathbf{HSP}(B_1) = \mathbf{HSP}(\mathbb{Z}). \tag{2}$$

Since B (being a domain) belongs to $\mathbf{HSP}(\mathbb{Q})$, so does each B_n. Thus, the classes in the second chain are all contained in $\mathbf{HSP}(\mathbb{Q})$.

Example. We exhibit a nil-4 disconnected f-ring that fails to belong to $\mathbf{HSP}(\mathbb{Q})$. Here is an f-ring law of $\mathbb{Q}$ that is not implied by the f-ring identities:

$$X \wedge Y \wedge Z \wedge (XZ - Y^2) \wedge (YZ - X^3) \wedge (X^2Y - Z^2) \leq 0.$$

This law was first mentioned (implicitly) in [E] and it is discussed in [M]. To see this is true in $\mathbb{Q}$, we attempt to construct a counterexample. Suppose x, y and z are (strictly) positive rational numbers and $xz < y^2$ and $yz < x^3$. If all these things are true, then $xyz^2 < x^3y^2$, and therefore $z^2 < x^2y$. Thus, no counterexample in $\mathbb{Q}$ is possible. We now give an example (not previously in the literature) of a disconnected f-ring whose additive group is $\mathbb{Z}^9$ in which this law fails. Let A be the set of formal sums $a_0 + a_3t^3 + \cdots a_{10}t^{10}$, with $a_i \in \mathbb{Z}$. Order A so that a sum is positive if one of the a_i is non-zero and the first non-zero a_i is positive. Select two integers $m > 1$ and $M > m^2$, and define a multiplication as follows:

$$t^i \cdot t^j = \begin{cases} t^{i+j}, & \text{if } i + j < 10 \text{ or } i = 5 = j; \\ Mt^{10}, & \text{if } i + j = 10 \text{ and } i \neq j; \\ 0, & \text{if } i + j > 10. \end{cases}$$

It is easy to check that with this multiplication A is a toring. Now, let $x = t^3$, $y = t^4$ and $z = mt^5$. Then

$$\begin{aligned} xz &= mt^8 > t^8 = y^2, \\ yz &= mt^9 > t^9 = x^3, \text{ and} \\ x^2y = t^6t^4 &= Mt^{10} > m^2t^{10} = z^2. \end{aligned}$$

Thus, the elements $x, y, z \in A$ violate the law above. Of course $\mathbb{Q}$ violates (D), so we see that $\mathbf{HSP}(\mathbb{Q})$ and $\mathbf{DFR}$ are incomparable. Proposition 3 implies that there is no equational class properly between $\mathbf{HSP}(\mathbb{Q})$ and $\mathbf{HSP}(\mathbb{Q}) \cap \mathbf{DFR}$. ∎

The example yields the following:

Proposition 4. *If $n > 3$, $\mathbf{D}_n\mathbf{FR}$ is incomparable with $\mathbf{HSP}(\mathbb{Q})$.* ∎

What is the relationship of $\mathbf{D}_n\mathbf{FR}$ to $\mathbf{HSP}(\mathbb{Q})$ for $n = 1, 2, 3$? It is clear that $\mathbf{D}_1\mathbf{FR}(= \mathbf{SFR})$ is contained in $\mathbf{HSP}(\mathbb{Q})$, since the torings in $\mathbf{SFR}$ are domains.

Proposition 5. $\mathbf{D}_2\mathbf{FR} \subset \mathbf{HSP}(\mathbb{Q})$.

Proof. Suppose that A is a finitely-generated toring in $\mathbf{D}_2\mathbf{FR}$. Let P be the positive cone of A and let N be the nilradical of A. By hypothesis, $N^2 = 0$, and, of course, N is prime. Let $A_N := \{\, a/b \mid a, b \in A \ \&\ b \notin N \,\}$ be the localization of A at N. This is a totally-ordered $\mathbb{Q}$-algebra with positive cone $\{\, a/b \mid a, b \in P \ \&\ b \notin N \,\}$. A_N contains A as a subring, so it suffices to show that $A_N \in \mathbf{HSP}(\mathbb{Q})$. Now, A_N/N is a totally-ordered field. By the structure theorem for complete local rings (see [ZS], page 304), A_N contains a subfield K that maps isomorphically onto A_N/N via the canonical projection $A_N \to A_N/N$. Moreover, N is a totally-ordered vector space over K of finite dimension. Let $e_1, \ldots, e_m$ be a basis. Consider the map of K-algebras: $K[X_1, \ldots, X_m] \to A_N; X_i \mapsto e_i$. It suffices to show that there is a total order on $K[X_1, \ldots, X_m]$ with respect to which this map is order-preserving. The argument given in [HI], in the proof of Theorem 3.10, shows that this is the case. ∎

The $n = 3$ case is open.

We will show that $\mathbf{HSP}(B)$ is *properly* contained in $\mathbf{DFR} \cap \mathbf{HSP}(\mathbb{Q})$ and that for all k, $\mathbf{HSP}(B_k)$ is *properly* contained in $\mathbf{D}_k\mathbf{FR} \cap \mathbf{HSP}(\mathbb{Q})$. We begin with an example that shows that $\mathbf{SFR} \neq \mathbf{HSP}(\mathbb{Z})$.

Example. Suppose that α is an algebraic irrational with minimum polynomial

$$f(X) = a_nX^n + a_{n-1}X^{n-1} + \cdots + a_0 \in \mathbb{Z}[X].$$

Then the following f-ring law is valid in $\mathbb{Z}$:

$$1 - |f(X)| \leq 0.$$

Let $f^*(X, Y) := a_nX^n + a_{n-1}X^{n-1}Y + \cdots + a_0Y^n$ be the homogenization of f. If $x, y \in \mathbb{Z}$, then $f^*(x, y) = 0$ if and only if $x = y = 0$. Thus, the following law is valid in $\mathbb{Z}$:

$$|X| + |Y| \wedge (1 - |f^*(X, Y)| \leq 0. \qquad (H_f)$$

Now suppose further that $\alpha \in \mathbb{R}_{>0}$. Let $\mathbb{R}[Y]$ be totally ordered as a ring so that $\lambda < Y$ for all $\lambda \in \mathbb{R}$. Consider the ordered subring $\mathbb{Z}[Y, \alpha Y] \subseteq \mathbb{R}[Y]$. Its elements can be written in the form: $g_0 + g_1(\alpha)Y + g_2(\alpha)Y^2 + \cdots g_m(\alpha)Y^m$, where g_i is a polynomial of degree i with integer coefficients. Since $\lambda Y^s < Y^{s+1}$ for all $s = 1, 2, \ldots$ and all $\lambda \in \mathbb{R}$, we see that there are no elements of $\mathbb{Z}[Y, \alpha Y]$ between 0 and 1. Thus, $\mathbb{Z}[Y, \alpha Y]$ is in **SFR**. However, $\mathbb{Z}[Y, \alpha Y]$ violates (H_f), for

$$\begin{aligned} |\alpha Y| + |Y| \wedge (1 - |f^*(\alpha Y, Y)|) &= (\alpha + 1)Y \wedge (1 - |f(\alpha)|Y^n) \\ &= (\alpha + 1)Y \wedge 1 = 1. \end{aligned}$$

Therefore, $\mathbb{Z}[Y, \alpha Y]$ is not in $\mathbf{HSP}(\mathbb{Z})$. ▪

If α and β are algebraic numbers with minimum polynomials f and g and $\mathbb{Q}(\alpha) \cap \mathbb{Q}(\beta) = \mathbb{Q}$, then $\mathbb{Z}[Y, \alpha Y]$ violates (H_f) but satisfies (H_g), while and $\mathbb{Z}[Y, \beta Y]$ violates (H_g) but satisfies (H_f). Thus, the **HSP** classes generated by these rings are incomparable. In particular, the classes $\mathbf{HSP}(\mathbb{Z}[Y\sqrt{p}Y])$, p a prime integer, are all incomparable, and they form an antichain in the interval between $\mathbf{HSP}(\mathbb{Z})$ and **SFR**.

The identity (H_f) fails in any f-ring that contains infinitesimals, for if x and y are infinitely small but non-zero, then $|f^*(x, y)| < 1$. We can repair this by means of the following law:

$$\bigl(2|X| + 2|Y| - 1\bigr) \wedge \bigl(1 - 2|f^*(X, Y)|\bigr) \leq 0. \qquad (H'_f)$$

This law is valid in B, for if either x or y is not infinitesimal, then $|f^*(x, y)| > 1$, while if both are infinitesimal, then $2|x| + 2|y| < 1$. On the other hand, it fails in $B[Y, \alpha Y]$ (Y infinitely large) for the same reason that (H_f) fails in $\mathbb{Z}[Y, \alpha Y]$. Since B is not in any of the varieties $\mathbf{D}_n\mathbf{FR}$, we have verified the claim about **DFR** in item *vi*) of the introduction. Considering B_n and $B_n[Y, \alpha Y]$ shows the rest of that item.

4. Open questions.

The following questions arose during this work, but we have not been able to answer them.

1. Is $\mathbf{D}_3\mathbf{FR} \subseteq \mathbf{HSP}(\mathbb{Q})$? We believe this to be the case.
2. Is $\mathbf{DFR} = \vee\{\mathbf{D}_n\mathbf{FR} \mid n \in \mathbb{N}\}$? We believe this to be the case.
3. Let $\mathbf{C}$ be an equational class of f-rings that contains $\mathbb{Q}$. Is $\mathbf{C}$ generated by $\mathbb{Q}$ together with $\mathbf{C} \cap \mathbf{DFR}$?

The following problem does not involve disconnected f-rings, but we believe it to be related to question 1 at a deep level:

4. Is it true that every f-ring identity in two variables implied by the identities defining f-rings? (By the theory of [HI], a positive answer to this question is equivalent to the assertion that the free f-ring on two generators is reduced. In a letter to the author dated February 24, 1997, Isbell suggested this might be true, but added, “No clue how to prove it.”)

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