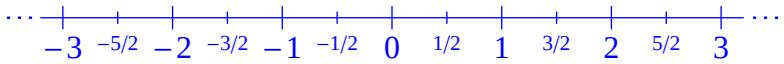
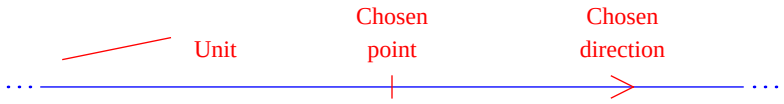
**Lecture 1. The Number Line**

The Number Line is the most important picture in mathematics.

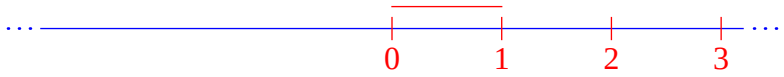



To make a number line, one decorates a naked line. Follow these steps:

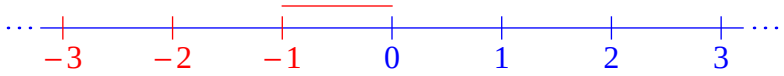
- STAGE I (setting things up)
  - Choose a unit of length.
  - Choose a point on the line.
  - Choose a direction from the point.



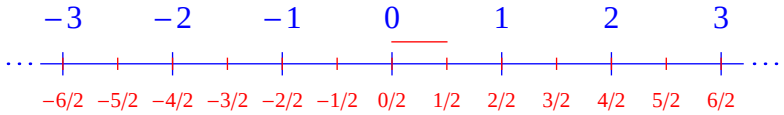

- STAGE II (labeling the integer points)
  - Lay copies of the unit end-to-end, starting from the chosen point and going in the chosen direction.
  - Label the endpoints of the units with the number of units one crosses to reach them.



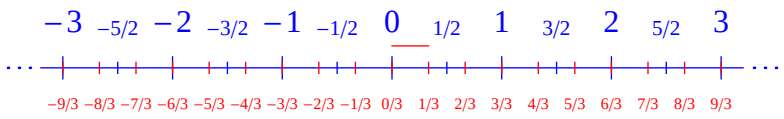

  - Lay copies of the unit, end-to-end, starting from the chosen point and going in the opposite of the chosen direction.
  - Label the endpoints of the units with the negative of the number of units one crosses to reach them.




- STAGE III (labeling rational points)
  - Divide the unit into 2 equal parts and use this to label the halves.



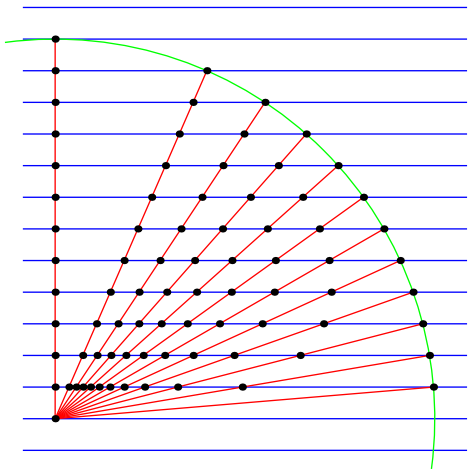

  - Divide the unit into 3 equal parts and use this to label the thirds.




  - Etc.

**How do you divide the unit into $n$ equal parts?**

- Draw more than $n$ parallel lines that are spaced so close that the unit can cross all of them.
- Put the one endpoint of the unit on the first line, and tilt it until the other endpoint lies on the $(n+1)^{th}$ line.



**Problems**

*Divide into groups of 3 to 5 people. Each group will consider one problem. Before talking, each participant will take 5 minutes to write and answer to his or her group's question. Discussion begins with each group member reading his/her written answer. After 10 more minutes, we will pick group representatives to report on the group's work to the class.*

1. Why do we need a common denominator when we add fractions?
2. Using the number line, show how we add 2/5 and 7/6. How does the common denominator 30 come into the picture?
3. Why don't we need a common denominator when we multiply fractions?
4. Use a rectangle to illustrate the meaning of multiplying 2/5 and 7/6.
5. When we divide one fraction by another, why do we invert and multiply?
6. Draw a picture to illustrate what it means to divide one fraction by another. (Compare with question 4.)

*Advanced discussion and questions for later in the day.* Stage II requires making infinitely many marks, so no person could ever complete it. Nonetheless, we can imagine a number line with all the integers labelled. Now, each substage of STAGE III requires infinitely many steps, but we can also imagine a number line with all the units and all the halves marked. If we can imagine to first two substages complete, the we can also imagine the first hundred substages complete.

1. Suppose I have marked all the multiples of 1/2, of 1/3, ... etc. and so on to 1/100. What is the smallest distance between neighboring marks?
2. I have marked all the multiples of 1/2, of 1/3, ... etc. and so on to 1/100. Suppose that I also mark all the positions that I can get to by adding and subtracting these positions. Now, what is the smallest distance between neighboring marks?
3. If we can imagine the first 100 substages complete, then we can imagine *all* the substages complete. After completing them all, can we create additional marks on the line by adding and subtracting existing marks? How about multiplying and dividing?
4. After completing STAGE III, are there any positions on the line that are not yet marked?