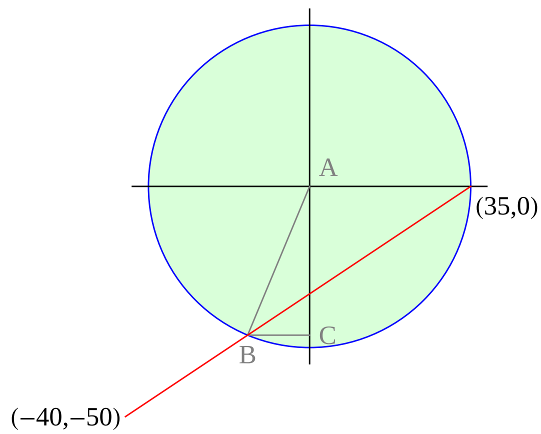
Lecture 10: Pythagoras, circles and lines June 15, 2010

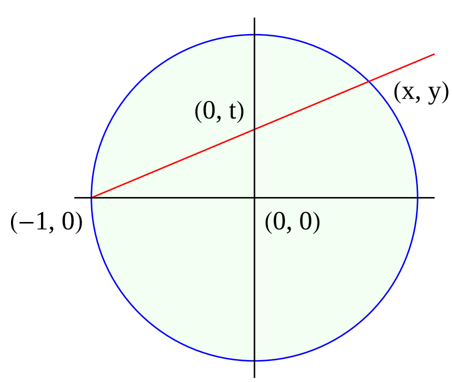
Post-lecture report

The Golf Problem. On Tuesday afternoon, we looked at Problem 4.8 of [Collingwood & Prince, *Precalculus*. Department of Mathematics, University of Washington, 2007]. In it, the cup on the 9^{th} hole of a golf course is located at the center of a circular green that is 70 feet in diameter. We set up a coordinate system with unit 1 foot, placing the origin at the cup and letting the axes run south to north and west to east. A ball located at the point $(-40, -50)$ follows a straight line path and exits the green at $(35, 0)$. We were to find the coordinates of the point B where the ball enters the green.



We were able to determine that B is $-k(5, 12)$, with $k = \frac{35}{13}$. Side AB has length $35 = 13\,k$, side AC has length $12\,k$ and side BC has length $5k$. So, there is a 5-12-13 right triangle hidden in the construction of the problem. The fact that the coordinates of B are rational is noteworthy. Perhaps the authors aimed for this in constructing the problem, since it is kind to students. The 5-12-13 triangle seems to be related to the rationality of the solution.

This raises an interesting question. If the ball were to start from a different point—(p, q), say—but still exit at $(35, 0)$, would the coordinates of the entry point still be rational. Clearly, if we let (p, q) vary arbitrarily, then any point on the circle could be an entry point. But what if we demand that p and q be integers? We speculated that even in this case, it might be that some choices of p and q lead to an entry point with rational coordinates, while other do not. We agreed that it would be surprising if it were easy to find a lot of integers for which B was rational. To investigate this, I suggested looking at a problem with nicer numbers:



Problem. Where does the line $y = t\,(x + 1)$ cross the circle $x^2 + y^2 = 1$?

Solution (Wednesday afternoon). We must solve two equations simultaneously. We can use the equation of the line to eliminate y from the equation of the circle. This gives:

$$x^2 + t^2(x+1)^2 = 1.$$

Now, we are trying to solve for x treating t as a constant. The equation has constant terms, terms in x and in x^2, so it is a quadratic. We will solve it with the quadratic formula. To do this, we must identify the coefficients. We can rewrite it

$$(t^2+1)\,x^2 + 2\,t^2\,x + (t^2-1) = 0.$$

Thus,

$$\begin{aligned} x &= \frac{-2t^2 \pm \sqrt{4t^4 - 4(t^2+1)(t^2-1)}}{2(t^2+1)} \\ &= \frac{-t^2 \pm 1}{t^2+1}, \end{aligned}$$

so,

$$x = -1 \qquad \text{or} \qquad x = \frac{1-t^2}{1+t^2}.$$

Using $y = t(x+1)$, we find the corresponding values of y to be

$$y = 0 \qquad \text{or} \qquad y = \frac{2\,t}{1+t^2}.$$

The solution $(x,y) = (-1,0)$ is the "pivot point" that is common to all the lines with equation $y = t(x+1)$. The solution

$$(x,y) = \left(\frac{1-t^2}{1+t^2}, \frac{2\,t}{1+t^2}\right) \tag{$*$}$$

is dependent upon t. But notice that if t is rational, then so are the coordinates of this point.

Problem. How does this apply to the original problem, in a circle of radius 35? (The easiest way to answer is by using some "proportional reasoning.")

If a **right triangle** has **integer** side lengths $a < b < c$, then (a,b,c) is called a *Pythagorean triple*. The triple $(3,4,5)$ appears in many algebra and calculus exercises; others are $(5,12,13)$, $(8,15,17)$, $(7,24,25)$, $(20,21,29)$, $(12,35,37)$, $(9,40,41)$. A Pythagorean triple (a,b,c) is said to be *primitive* if the greatest common divisor of a, b and c is 1.

Fact. If (a,b,c) is a Pythagorean triple, then $\left(\frac{a}{c}, \frac{b}{c}\right)$ is a point on the circle $x^2+y^2=1$.

Fact. Every point on the circle $x^2+y^2=1$ with *both* coordinates rational is associated with a primitive Pythagorean triple.

Reason. Suppose (q_1,q_2) is a rational point on the unit circle. Write q_1 and q_2 with a common denominator: $q_1 = \frac{a}{c}$, $q_2 = \frac{b}{c}$. If we are not careful, we may find that the a, b and c we have chosen have a common factor, but if so, divide it out and use the numbers that result. Since $\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$, $a^2+b^2=c^2$. So (a,b,c) is a primitive Pythagorean triple.

Fact. If we set $t = \frac{n}{m}$ in $(*)$, then $(x,y) = \left(\frac{m^2-n^2}{m^2+n^2}, \frac{2mn}{m^2+n^2}\right)$, and we get the Pythagorean triple $(m^2-n^2, 2mn, m^2+n^2)$. (See CCSS, page 64: *"Prove polynomial identities and use them to describe numerical relationships. For example, the polynomial identity $(x^2+y^2)^2 = (x^2-y^2)^2 + (2xy)^2$ can be used to generate Pythagorean triples."*)