Lecture 11: Slopes and Right Angles June 16, 2010

We now introduce the idea of the *slope of a segment.* The slope is always measured relative to a given coordinate system, but as long as we have only one coordinate system at hand, there is no reason to keep on mentioning it.

Definition. If AC is a segment and $A = (a, b)$ and $C = (c, d)$, then *the slope of* AC is

$$\frac{d-b}{c-a}.$$

This is the familiar "rise over the run." Notice that a vertical segment does not have a slope. The segment AC has the same slope as the segment CA. It does not matter in what order the endpoints are given.

Comment. It is true that any two segments that are contained in the same line have the same slope; we shall return to this important fact later. Unlike the definition of the slope of a segment, this fact is NOT a mere matter of definition. It is a wonderful consequence of the definition of the slope of a segment and the properties of the cartesian coordinate system.

Fact 1. If we translate a segment, its slope does not change.

Reason. If $A' = (a, b) + (p, q)$ and $C' = (c, d) + (p, q)$, then the slope of $A'C'$ is the same as the slope of AC.

We can recognize a right triangle by examining the slopes of its legs.

Fact 2. In a triangle $\triangle ACB$, $\angle C$ is right if and only if the product of the slopes of AC and BC is -1 or one of these segments is parallel to the x-axis the the other is parallel to the y-axis.

Reason. The previous fact shows that if we can present the reasons for this assertion when $C = (0, 0)$, then the reasons in general will be understood. Suppose $C = (0, 0)$, $A = (p, q)$ and $B = (r, s)$, with p and r not equal to zero. The slope of segment CA is q/p and the slope of segment CB is s/r.

$$\begin{aligned}
\angle C \text{ is right} \quad &\iff \quad |AC|^2 + |BC|^2 = |AB|^2 \qquad \text{(Pythagorean Theorem and Its Converse)}\\
&\iff \quad p^2 + q^2 + r^2 + s^2 = (p-r)^2 + (q-s)^2\\
&\iff \quad p^2 + q^2 + r^2 + s^2 = p^2 - 2pr + r^2 + q^2 - 2qs + s^2\\
&\iff \quad 0 = -2pr - 2qs\\
&\iff \quad \frac{q}{p} = -\frac{s}{r}\\
&\iff \quad (\text{slope of } AC) = (-1)\ (\text{slope of } BC).
\end{aligned}$$

The Pythagorean Theorem and Its Converse:
Suppose $\triangle ABC$ has sides of length a, b and c, as shown. Then

$$\angle C \text{ is right} \quad \text{if and only if} \quad a^2 + b^2 = c^2.$$