Lecture 12: Changing coordinates and transforming graphs June 17, 2010

Introduction

Suppose $\Theta(x,y)$ is a statement in which the variables x and y appear. For example, $\Theta(x,y)$ might be the statement $x^2+y^2=35^2$. It could also be a more complicated statement, such as:

$$x^2+y^2=35^2 \quad \text{and} \quad y=\frac{2}{3}(x+1),$$

or

$$x^2+y^2=35^2 \quad \text{and } x \text{ and } y \text{ are both rational.}$$

No matter how complicated $\Theta(x,y)$ is, by the *graph of* $\Theta(x,y)$, we mean the set of all number pairs for which $\Theta(x,y)$ is true. If we interpret number pairs as points by referring to the standard x-y-coordinate system on the plane, then the graph of $\Theta(x,y)$ is a set of points in the plane.[1]

We will use the symbol X to denote the graph:

$$X := \{\,(x,y) \mid \Theta(x,y)\,\}.$$

Now, suppose a and b are any numbers. Consider the set

$$X' := \{\,(x,y) \mid \Theta(x-a,y-b)\,\}.$$

I assert that X' coincides with the set of all pairs obtained by taking a pair belonging to X and adding (a,b) to it. In other words, X' is the image of X "after a translation by (a,b)." We can demonstrate this assertion as follows:

$$\begin{aligned}\big((x_0+a),(y_0+b)\big)\in X' &\iff \Theta\big((x_0+a)-a,(y_0+b)-b\big)\\ &\iff \Theta(x_0,y_0)\\ &\iff (x_0,y_0)\in X\end{aligned}$$

Returning to the example of the circle, X' is the graph of $(x-a)^2+(y-b)^2=35^2$, which is the equation of the circle of radius 35 about (a,b). Clearly, this is the translation of the original circle by (a,b).

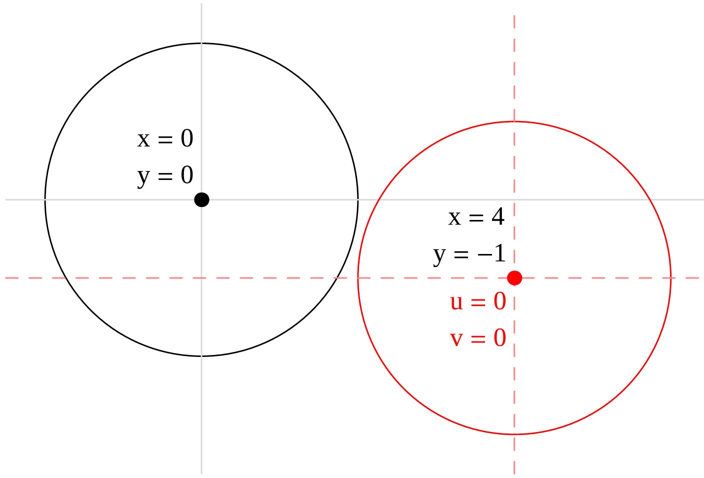
Introducing a new coordinate system

If we use the same unit of distance as in the x-y-coordinate system and set up a new coordinate system (u,v) with center at the point P where $x=a$ and $y=b$ and with axes parallel to those of

[1] What we have done here is taken a term that is familiar in high school math—the word "graph"—and pointed out that without changing its meaning, it can be used in a much broader set of circumstances than those that routinely appear in high school. The ideas here are no more complicated than those that are used routinely in high school math, provided that the vocabulary of high school math is truly understood. Unfortunately, many high school textbooks foster subtle misinterpretations, or fail to connect the ideas that appear in definitions to practice. For example, while students learn that the graph of $y=mx+b$ is the line through $(0,b)$ with slope m, they often forget that the equation is a "point-tester" and view it only as code for, "b is the y-intercept and m is the slope." Some textbooks and curricula aggravate the problem by inundating students with exercises that emphasize decoding while letting the meaning fade.

the (x, y)-system, then $u = x - a$ and $v = y - b$. The equation of the circle of radius r about P is $u^2 + v^2 = r^2$ in the u-v-system.

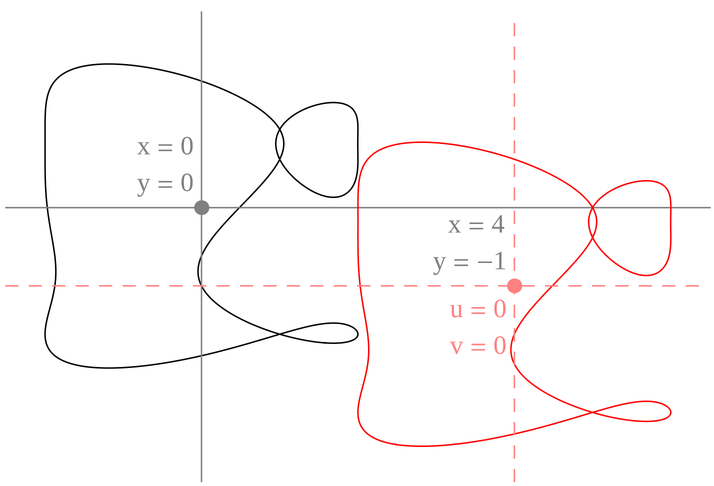
Example 1. Here, we illustrate this with $a = 4$ and $b = -1$.



On the left, in black, is the graph of $x^2 + y^2 = 2^2$. The x-y-axes are shown in light gray. On the right, in red, is the graph of $(x - 4)^2 + (y + 1)^2 = 2^2$. (Note that $y + 1 = y - (-1)$.) The u-v-coordinate system has its origin at the point where $x = 4$ and $y = -1$. The u-v-axes are shown in dashed pink. Note that $u = x - 4$ and $v = y - (-1) = y + 1$. In u-v-coordinates, the equation of the righthand circle is $u^2 + v^2 = 2^2$. /////

Summary idea: if u and v are new coordinates such that $u = x - a$ and $v = y - b$, then they define a coordinate system that is just like the x-y-system except for the position of its origin. The u-v-origin is at the point where $x = a$ and $y = b$.

Example 2.



On the left, in black, is the graph of

$$(x^2 - 4)(x - 1)^2 + (y^3 - 2y + 1)^2 = 0,$$

or in expanded form,

$$y^6 + x^4 - 4y^4 - 2x^3 + 2y^3 - 3x^2 + 4y^2 + 8x - 4y - 3 = 0.$$

On the right, in red, is the graph of

$$((x-4)^2-4)((x-4)-1)^2+((y+1)^3-2(y+1)+1)^2=0,$$

or in expanded form

$$y^6+6y^5+x^4+11y^4-18x^3+6y^3+117x^2+y^2-320x+300=0.$$

Notice that in expanded form, the close relationship between the two equations is quite hidden. As before, the u-v-coordinate system has its origin at the point where $x=4$ and $y=-1$; the u-v-axes are shown in dashed pink; $u=x-4$ and $v=y-(-1)=y+1$. In u-v-coordinates, the equation of the righthand figure is

$$(u^2-4)(u-1)^2+(v^3-2v+1)^2=0.$$

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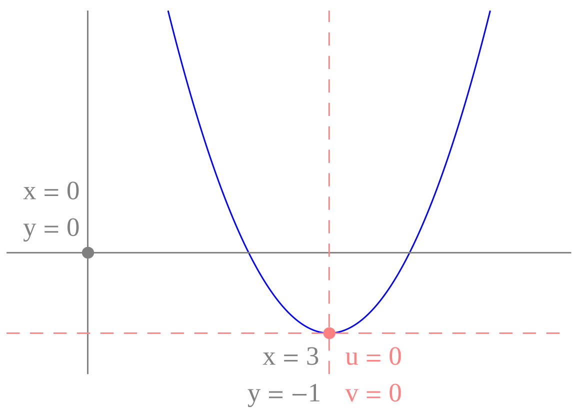
Using new coordinates to simplify the equation for a figure

Example 3. In the previous examples, we started with a graph, then made a new graph by translating. In the process, we introduced a new coordinate system. In this example, we work with only one figure. We show that by introducing a new coordinate system, we can describe the figure with a simpler equation.



In blue, we have the graph of $y=x^2-6x+8$. We have overlaid the coordinate axes of a u-v-coordinate system with origin at $x=3$ and $y=-1$, so $u=x-3$ and $v=y+1$. In this system, the blue parabola is given by the equation $v=u^2$. We can verify this by noting that

$$\begin{aligned} v=u^2 &\Leftrightarrow (y+1)=(x-3)^2 \\ &\Leftrightarrow y+1=x^2-6x+9 \\ &\Leftrightarrow y=x^2-6x+8 \end{aligned}$$