Lecture 13 (supplement)

These notes exhibit a few ways of showing that:

a) every line in the plane is the graph of a linear equation, and

a) the graph of every linear equation is a line?

The first demonstration, which is based on similar triangles, follows a pattern that can be found in expositions of analytic geometry in the early 18th century. I have found this argument repeated in algebra textbooks from the 1970s. The fat Algebra textbook of Sullivan, which was formerly used for Math 1021 at LSU, has a version of this, but it's not presented carefully, and there is much hand-waving.

Theorem 1. *Suppose that $P_0 = (x_0, y_0)$ and $P_1 = (x_1, y_1)$ are points in the plane with $x_0 \neq x_1$. Let L be the line through P_0 and P_1, and let $m = \frac{y_1 - y_0}{x_1 - x_0}$. Then, for any point $P = (x, y)$ in the plane:*

$$P \text{ is on } L$$

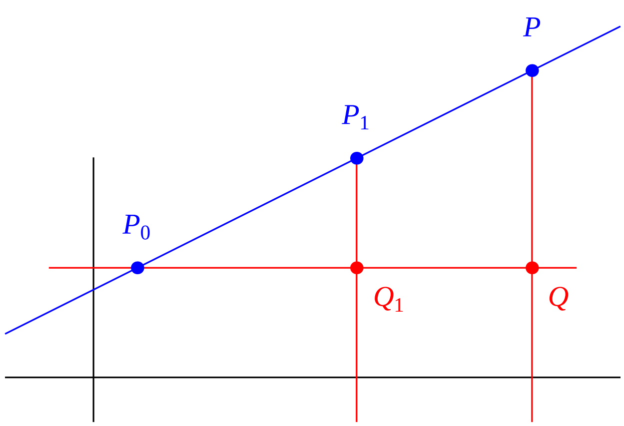
if and only if

$$y - y_0 = m(x - x_0). \tag{1}$$

Proof. If $y_1 = y_0$, then equation (1) reads $y = y_0$ and L is the line perpendicular to the y-axis through $(0, y_0)$. The conclusion in this case is immediate from the definition of the coordinate system. Let us turn to the situation when $y_1 \neq y_0$. Let $Q_1 = (x_1, y_0)$ and let $Q = (x, y_0)$.

($\Rightarrow$) Suppose $P = (x, y)$ is on L. We will show that the coordinates of P satisfy (1).

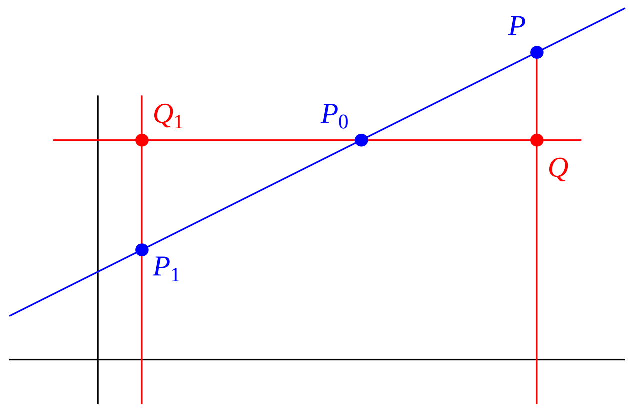
Case 1: If $P = P_1$, then (1) is clearly satisfied. Suppose P_1 and P are on the same side of the vertical line through P_0.



In this case, rays $\vec{P_0P_1}$ and $\vec{P_0P}$ are the same and rays $\vec{P_0Q_1}$ and $\vec{P_0Q}$ are also the same. Therefore, right triangles $\triangle P_0Q_1P_1$ and $\triangle P_0QP$ are similar, since they share an angle. Moreover, the sign of $x_1 - x_0$ is the same as the sign of $x - x_0$ and the sign of $y_1 - y_0$ is the same as the sign of $y - y_0$. Since corresponding sides of similar triangles are proportional, we see that

$$\frac{y - y_0}{x - x_0} = \frac{y_1 - y_0}{x_1 - x_0} = m.$$

Case 2: P_1 and P_2 are on opposite sides of the vertical line through P_0.



In this case, $\vec{P_0P_1}$ and $\vec{P_0P}$ are opposite colinear rays and so are $\vec{P_0Q_1}$ and $\vec{P_0Q}$. Therefore, right triangles $\triangle P_0Q_1P_1$ and $\triangle P_0QP$ are similar, since the angles at P_0 are vertical. Moreover, the sign of $x_1 - x_0$ is opposite the sign of $x - x_0$ and the sign of $y_1 - y_0$ is opposite the sign of $y - y_0$. This implies that

$$\frac{y - y_0}{x - x_0} = \frac{y_1 - y_0}{x_1 - x_0} = m.$$

($\Leftarrow$) Suppose P is *not* on L. Let $P' = (x, y')$ be the point of intersection of L and the vertical line through P. Then by the first part of the proof,

$$y' - y_0 = m(x - x_0).$$

Since $y' \neq y$,

$$y - y_0 \neq m(x - x_0). \qquad /////$$

A second approach

Another version proceeds by first proving a simple version of the previous theorem, and then using change of coordinates.

Lemma 1. *Let L any line through $O = (0, 0)$, and $P_1 = (x_1, y_1)$ $(x_1 \neq 0)$, and let $m = \frac{y_1}{x_1}$. Then, for any point $P = (x, y)$ in the plane:*

$$P \text{ is on } L$$

if and only if

$$y = m\,x. \tag{1}$$

Proof. The proof of Lemma 1 is exactly like the proof of Theorem 1, but it is a lot simpler because Q and Q_1 lie on the x-axis. /////

Proof of Theorem 1. We are going to use a change of coordinates, so we will use the notation $x(P)$ and $y(P)$ for the x and y coordinates of P and $t(P)$ and $u(P)$ for the coordinates of P in a system with coordinate functions t and u. Choose a t-u-coordinate system centered at P_0, which is a translate of the x-y-system. Then $t(P) = x(P) - x_0$ and $u(P) = y(P) - y_0$. Let $m = u(P_1)/t(P_1)$. By Lemma 1, $P \in L$ iff $u(P) = m\,t(P)$. But the latter equation is equivalent to $y(P) - y_0 = m(x(P) - x_0)$. /////

A third approach

The approach does not depend on similar triangles, but on the following fact:

Lemma 2. *Let $Q = (a, b)$ and $Z = (c, d)$ be two points in the plane, both different from the origin $O = (0, 0)$. Then $\angle QOZ$ is right if and only if $0 = ac + bd$.*

Proof. By the Pythagorean Theorem and its converse,

$$\begin{aligned} \angle QOZ \text{ is right} &\Leftrightarrow |OQ|^2 + |OZ|^2 = |QZ|^2 \\ &\Leftrightarrow a^2 + b^2 + c^2 + d^2 = (c-a)^2 + (d-b)^2 \\ &\Leftrightarrow a^2 + b^2 + c^2 + d^2 = c^2 - 2ac + a^2 + (d^2 - 2bd) + b^2 \\ &\Leftrightarrow 0 = -2ac - 2bd \\ &\Leftrightarrow 0 = ac + bd. \qquad ///// \end{aligned}$$

Suppose an x-y-coordinate system is given. Let L be any line and P_0 be any point on it. Choose t-u coordinates centered at P_0, so $t(P) = x(P) - x(P_0)$ and $u(P) = y(P) - y(P_0)$ Choose Q so that QP_0 is perpendicular to L. Let $a := t(Q)$ and $b := u(Q)$. Then

$$\begin{aligned} P \in L &\Leftrightarrow P = P_0 \text{ or } \angle QP_0P \text{ is right} \\ &\Leftrightarrow 0 = a\,t(P) + b\,u(P) \\ &\Leftrightarrow 0 = a\,(x(P) - x(P_0)) + b(y(P) - y(P_0) \\ &\Leftrightarrow a\,x(P_0) + b\,y(P_0) = a\,x(P) + b\,y(P). \end{aligned}$$

Thus, $P \in L$ if and only if the x-y-coordinates of P satisfy

$$c = a\,x + b\,y,$$

where $c = a\,x_0 + b\,y_0$, with $P_0 = (x_0, y_0)$ on L and (a, b) as above.