Lecture 13: More about lines June 20, 2010

We have shown that every line is the graph of a linear equation (Lecture 9, page 3, Problem 2). We did this by recognizing that given any line, we can find a segment AB of which that line is the perpendicular bisector. The line then consists of those points equidistant from A and B. Using the distance formula, the set of points equidistant from A and B can be described by an equation. That equation simplifies to a linear one.

We did not show yet that the graph of any linear equation is a line. (Problem 3 from Lecture 9 provides one way to do this, but we didn't do that problem.) This is something so basic and so well-known that you may wonder why we should even bother to ask for an explanation. (You might be wondering, "What will he ask next? Why do we bathe? Why do we dress ourselves?")

The *Common Core Standards* for eighth grade includes the statement that students should:

> *Use similar triangles to explain why the slope m is the same between any two distinct points on a non-vertical line in the coordinate plane; derive the equation $y = mx$ for a line through the origin and the equation $y = mx + b$ for a line intercepting the vertical axis at b.*

Interestingly, in the latest issue of the *Notices of the American Mathematical Society* there is an article by a mathematics educator and a mathematician that is highly relevant. The authors reviewed several high school textbooks, and commented on what they found. They state:

> *Understanding linear functions is fundamental to a good Algebra 1 course. The connection between the graph of a linear function and the algebraic version is important. We were disappointed [with the textbooks we reviewed]. No program produced the basics here. Slope, although defined, is never shown to be well-defined. It is never shown that the graph of an algebraic linear function really is a line in the coordinate plane, and it is never shown that a line in the coordinate plane really is the graph of an algebraic linear function. The worst aspect of this was that it seemed the textbook authors were unaware that something was missing.* [Guershon Harel and W. Stephen Wilson. The State of High School Textbooks. *Notices of the AMS*, 58(6):823-826, June/July 2011.]

Delivering the goods

As in Lecture 9, in order to show that the graph of a linear equation is a line, we need a description of lines that we can connect with equations. This time, we need to start with the equation. Well, it turns out that we produced the right tool in Lecture 11. There, we defined the slope of a segment, and using that, we showed that the angle between two segments was right if and only if the product of the slopes was -1. This can be stated another way:

Proposition. Suppose $P = (a, b)$ is a point other than $O = (0, 0)$ and $Z = (x, y)$ is another point in the plane. Then the following are equivalent:
a) PO and OZ are perpendicular;
b) $a\,x + b\,y = 0$;
c) $y/x = -a/b$.

Now we shall use these facts to show that the graph of any equation of the form

$$a\,x + b\,y = 0 \quad \text{with at least one of } a \text{ and } b \text{ nonzero}$$

is a line.

Let P be a point other than $O = (0, 0)$ and let $Z = (x, y)$ be any other point in the plane. The set of all Z such that PO is perpendicular to OZ together with the point O itself is clearly a line. In symbols, we would write:

$$\{\, Z \mid PO \perp OZ \,\} \cup \{O\} \quad \text{is a line.}$$

Since $Z = (x, y)$, and since $a\,x + b\,y = 0$ means that $PO \perp OZ$ (or $Z = O$),

$$\{\, (x, y) \mid a\,x + b\,y = 0 \,\} \quad \text{is a line.}$$

Finally, since P could have been any point other than $(0, 0)$, a and b could have been any constants not both zero. So this shows that the graph of *any* equation of the form $a\,x + b\,y = 0$ is a line.

Challenge Problem. Generalize this argument to show that the graph of $a\,x + b\,y = c$ is also a line. Obviously, to do this, you really need to understand the argument above. So, the analysis of this should be your first goal.

I realize, of course, that the kind of thinking and analysis I am asking you to engage in is not very closely connected to the kinds of things you are likely to be doing in your classrooms at this time. And it is not my intention to take you so far from the challenges that you face that you cannot see some connections. If we are losing that connection, then let me know. But, on the other hand, as our psychologist friends are telling us, I do want to create a real set of meaningful challenges. I think the Standards themselves, and the words of some math educators, show that challenges of the kind I am making are indeed meaningful. And I feel that you are fully smart enough and dedicated enough to meet them. We will work together, and we will learn much from each other.