Lecture 14: Functions June 21, 2010

A set is a collection of things. All mathematical objects are sets. Usually, they have some additional structure. One kind of structure that we find in many of the sets that occur in mathematics is the presence of *operations*, such as $+$ or $\times$. Another kind of structure is the *relation* of order $\leq$.

Set theory is a branch of mathematics that was invented somewhat over 100 years ago. The purpose of set theory is to study the properties of sets apart from any additional structure. Two sets are *equivalent* (from the point of view of set theory) if they can be put into a one-to-one correspondence, with every element of the first set corresponding with exactly one element of the second, and every element of the second set "receiving correspondence" from exactly one element of the first. One of the first great observations of set theory was that the set of rational numbers $\mathbb{Q}$ is equivalent to the set of integers $\mathbb{Z}$, but not to the set of real numbers $\mathbb{Z}$.

Set theory is a truly rarified domain of thought, yet remarkably, mathematicians of the early 20th century showed that all of mathematics can be built out of sets. That is, every mathematical object can be modeled exactly by a construction with sets. The "New Math" of the 1960s was partly an attempt to bring this insight into school math. It didn't work out very well for most people. There were some students who thrived on it, however. Many of them chose careers in math.

In set theory, we study constructions that can be made from sets. Some of the most important are

- union: $A \cup B = \{\, x \mid x \in A \text{ or } x \in B \,\}$
- product: $A \times B = \{\, (x, y) \mid x \in A \text{ and } y \in B \,\}$
- exponentiation $B^A = \{\, f \mid f \text{ is a function from } A \text{ to } B \,\}$

We will explain these in more detail below.

Relations

If A and B are sets, a relation between A and B is a subset of $A \times B$.

Example. Suppose $A = B = \{1, 2, 3, 4, 5, 6\}$. Then $\{\, (x, y) \in A \times B \mid x \text{ is a factor of } y \,\}$ is illustrated below:

$$\begin{array}{ccccccccc}
 & & & & B & & & \\
 & & 1 & 2 & 3 & 4 & 5 & 6 \\
 & 1 & * & * & * & * & * & * \\
 & 2 & & * & & * & & * \\
A & 3 & & & * & & & * \\
 & 4 & & & & * & & \\
 & 5 & & & & & * & \\
 & 6 & & & & & & *
\end{array}$$

Example. $\{\, (x, y) \mid x^2 + y^2 = 1 \,\}$ (the unit circle) is a relation between $\mathbb{R}$ and $\mathbb{R}$.

Caution. MML seems to demand that in a relation between A and B, every element of A should occur as the first element of at least one pair in the relation. (Humpty Dumpty would not disapprove.)

Functions

If A and B are sets, a function f from A to B is a subset of $A \times B$ such that every element of A is the first element of exactly one element of f.

Problems

If A is a finite set, $|A|$ denotes the number of elements of A. For example, $|\{a, b, c\}| = 3$.

1. Show that if A and B are finite, then $|A \times B| = |A| \cdot |B|$.
2. Show that if A and B are finite, then $|B^A| = |B|^{|A|}$.
3. Suppose A, B and C are any sets. Find a one-to-one correspondence between $(A^B)^C$ and $A^{(B \times C)}$.