From: SMSG: Elementary Functions (Unit 21), Stanford University, 1965.

rately, the three possibilities $x < 0$, $0 \leq x < 2$, and $x \geq 2$.)

9. If $0 < x < 1$, we can multiply both sides of the inequality $x < 1$ by the positive number x to obtain $x^2 < x$, and we can similarly show that $x^3 < x^2$, $x^4 < x^3$, and so on. Use this result to show that if $|x| < 1$, then $|x^2 + 2x| < 3|x|$.

10. Show that, if $0 < x < k$, then $x^2 < kx$. Hence show that, if $|x| < 0.1$, then $|x^2 - 3x| < 3.1|x|$.

11. For what values of x is it true that $|x^2 + 2x| < 2.001|x|$?

1-5. Composition of Functions

Our consideration of functions, to this point, has been concerned with individual functions, with their domains and ranges, and with their graphs. We now consider certain things that can be done with two or more functions somewhat as, when we start school, we first learn numbers and then learn how to combine them in various ways. There is, as a matter of fact, a whole algebra of functions, just as there is an algebra of numbers. Functions can be added, subtracted, multiplied, and divided. The sum of two functions f and g, for example, is defined to be the function

$$f+g: x \longrightarrow f(x) + g(x)$$

which has for domain the intersection of the domains of f and g; there are similar definitions, which you can probably supply yourself, for the difference, product, and quotient of two functions. Because, for example, the number $(f+g)(x)$ can be found by adding the numbers $f(x)$ and $g(x)$, it follows that this part of the algebra of functions is so much like the familiar algebra of numbers that it would not pay us to examine it carefully. There is, however, one important operation in this algebra of functions that has no counterpart in the algebra of numbers: the operation of composition.

The basic idea of composition of two functions is that of a kind of "chain reaction" in which the functions occur one after the other. Thus, an automobile driver knows that the amount he depresses the accelerator pedal controls the amount of gasoline fed to the cylinders and this in turn affects the speed of the car. Again,

the momentum of a rocket sled when it is near the end of its ru depends on the velocity of the sled, and this in turn depends o the thrust of the propelling rockets.

Let us look at a specific illustration. Suppose that f the function $x \longrightarrow 3x - 1$ (this might be a time-velocity functio and suppose that g is the function $x \longrightarrow 2x^2$ (this might be a locity-energy function). Let us follow what happens when we "a these two functions in succession--first f, then g--to a part cular number, say the number 4. In brief, let us first calcula $f(4)$ and then calculate $g(f(4))$. (Read this "g of f of 4".)

First calculate $f(4)$. Since f is the function $x \longrightarrow 3x$ - $f(4) = 3 \cdot 4 - 1 = 11$. Then calculate $g(f(4))$, or $g(11)$. Since is the function $x \longrightarrow 2x^2$, $g(11) = 2 \cdot 11^2 = 242$. Thus $g(f(4))$ = $g(11) = 242$. In general, $g(f(x))$ is the result we obtain when first "apply" f to an element x and then "apply" g to the result. The function $x \longrightarrow g(f(x))$ is then called a composite of f and g, and denoted gf.

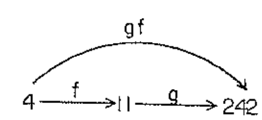
We say a composite rather than the composite because the or in which these functions occur is important. To see that this the case, start with the number 4 again, but this time find $g(4)$ first, then $f(g(4))$. The results are as follows:

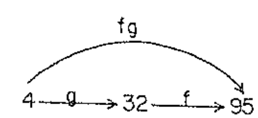
$$g(4) = 2 \cdot 4^2 = 32 \text{ and } f(g(4)) = f(32) = 3 \cdot 32 - 1 = 95.$$

Clearly $g(f(4))$, which is 242, is not the same as $f(g(4))$, which is 95.

Warning. When we write "gf" we mean that f is to be applie before g and then g is applied to $f(x)$. Since "f" is writte after "g" is written, this can easily lead to confusion. You ca avoid the confusion by thinking of the equation $(gf)(x) = g(f(x))$

It may be helpful to diagram the above process as follows: If gf is the function $x \longrightarrow g(f(x))$ and fg is the function $x \longrightarrow f(g(x))$ we have





Note particularly that fg is not the product of f and g mentioned earlier in this section. When we want to talk about this product, $f \cdot g$, we shall always use the dot as shown. Incidentally, for the above example, we have $(f \cdot g)(4) = f(4) \cdot g(4) = 11 \cdot 32 = 352 = 32 \cdot 11 = g(4) \cdot f(4) = (g \cdot f)(4)$.

To generalize this illustration, let us use x instead of 4 and find algebraic expressions for $(gf)(x)$ and $(fg)(x)$. We do this as follows:

$$(gf)(x) = g(f(x)) = g(3x - 1) = 2\cdot(3x - 1)^2$$

and $$(fg)(x) = f(g(x)) = f(2x^2) = 3(2x^2) - 1 = 6x^2 - 1.$$

Again, note that $(gf)(x)$ and $(fg)(x)$ are not the same so the function gf is not the same as the function fg. In symbols, $gf \neq fg$. If, now, we substitute 4 for x we obtain

$$(gf)(4) = 2(3 \cdot 4 - 1)^2 = 242$$

and $$(fg)(4) = 6 \cdot 4^2 - 1 = 95$$

These results agree with the ones we obtained above.

We are now ready to define the general process that we have been illustrating.

<u>Definition 1-7</u>. Given two functions, f and g, the function $x \longrightarrow g(f(x))$ is called a <u>composite</u> of f and g and denoted gf. The domain of gf is the set of all elements x in the domain of f for which $f(x)$ is in the domain of g. The operation of forming a composite of two functions is called <u>composition</u>.

<u>Example 1</u>. Given that $f: x \longrightarrow 3x - 2$ and $g: x \longrightarrow x^5$ for all $x \in R$, find

a) $(gf)(x)$
b) $(ff)(x)$
c) $f(g(x) + 3)$
d) $f(g(x) - f(x))$

<u>Solution</u>:

a) $(gf)(x) = g(f(x)) = g(3x - 2) = (3x - 2)^5$

b) $(ff)(x) = f(f(x)) = f(3x - 2) = 3(3x - 2) - 2 = 9x - 8$

c) $f(g(x) + 3) = f(x^5 + 3) = 3(x^5 + 3) - 2 = 3x^5 + 7$

d) $f(g(x) - f(x)) = f(x^5 - 3x + 2) = 3(x^5 - 3x + 2) - 2 = 3x^5 - 9x + 4$

If we think of a function as a machine with an input and an output, as suggested in Section 1-1, we see that two such machines can be arranged in tandem, so that the output of the first machine

feeds into the input of the second. This results in a "compo process that is analogous to the operation of composition. I illustrated in Figure 1-5b. In this figure the machine for the machine for g have been housed in one cabinet. This co machine is the machine for gf.

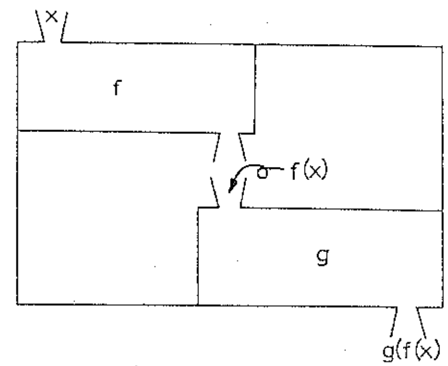


Figure 1-5b.
Schematic representation of the composition of functions

Note that the machine for gf will jam if either of two t happens:

a) It will jam if a number not in the domain of f is into the machine.

b) It will jam if the output $f(x)$ of f is not in the domain of g.

Thus, once again we see that the domain of gf is the set of a elements x in the domain of f for which $f(x)$ is in the do of g.

We have noted that the operation of composition is not c tative; that is, it is not always true that $fg = gf$. On the hand, it <u>is</u> true that this operation is associative: for any three functions f, g, and h, it is always true that $(fg)h$ $f(gh)$. We shall not prove this theorem; we shall however, il trate its operation by an example.

<u>Example 2</u>. Given $f: x \longrightarrow x^2 + x + 1$, $g: x \longrightarrow x + 2$, and $h: x \longrightarrow -2x - 3$, find

a) fg
b) gh
c) $(fg)h$
d) $f(gh)$

Solution:

a) $(fg)(x) = (x + 2)^2 + (x + 2) + 1 = x^2 + 5x + 7$, so $fg: x \longrightarrow x^2 + 5x + 7$

b) $(gh)(x) = (-2x - 3) + 2 = -2x - 1$, so $gh: x \longrightarrow -2x - 1$

c) $(fg)h: x \longrightarrow (-2x - 3)^2 + 5(-2x - 3) + 7$

d) $f(gh): x \longrightarrow (-2x - 1)^2 + (-2x - 1) + 1$

It is not altogether obvious from these expressions that $(fg)h$ and $f(gh)$ are the same function. But if you will simplify the expressions you will see that they are indeed the same.

Exercises 1-5

1. Given that $f: x \longrightarrow x^2 - 1$ and $g: x \longrightarrow x + 2$ for all $x \in R$, find
   a) $(fg)(-2)$
   b) $(gf)(0)$
   c) $(gg)(1)$
   d) $(ffg)(1)$
   e) $(fg)(x)$
   f) $(gf)(x)$
   g) $\dfrac{(fg)(x) - (fg)(1)}{x - 1}$ if $x \neq 1$

2. Let it be given that $f: x \longrightarrow ax + b$ and $g: x \longrightarrow cx + d$ for all $x \in R$.
   a) Find $(fg)(x)$.
   b) Find $(gf)(x)$.
   c) Compare the slopes of fg and gf with the slopes of f and g.
   d) Formulate a theorem concerning the slope of a composite of two linear functions.

3. Suppose that $f: x \longrightarrow 1/x$ for all real numbers x different from zero.
   a) Find $(ff)(1)$, $(ff)(-3)$, and $(ff)(8)$.
   b) Describe ff completely.

4. Let it be given that $j: x \longrightarrow x$ and $f: x \longrightarrow x + 2$ for all $x \in R$.
   a) Find fj and jf. [First find $(fj)(x)$ for all $x \in R$.]

   b) Find a function g such that $fg = j$. [That is, fin[d g] such that $(fg)(x) = j(x)$ for all $x \in R$.]
   c) Find a function h such that $hf = j$. Compare your [result] with that of (b).

5. a) If $f: x \longrightarrow x^2$ and $g: x \longrightarrow x^3$, find expressions for ([$fg)(x)$] and $(gf)(x)$.
   b) If $f: x \longrightarrow x^m$ and $g: x \longrightarrow x^n$, find expressions for ([$fg)(x)$] and $(gf)(x)$.

6. a) If $f: x \longrightarrow x^2$ and $g: x \longrightarrow x^3$, find an expression for [$(f \cdot g)$] (x), where $f \cdot g$ is the product of f and g; that i[s,] $(f \cdot g)(x) = f(x) \cdot g(x)$. Compare with Exercise 5(a).
   b) If $f: x \longrightarrow x^m$ and $g: x \longrightarrow x^n$ for all $x \in R$ (where m [and] n are positive integers), find an expression for (f[$\cdot g)(x)$]. Compare with Exercise 5(b).

7. Suppose that $f: x \longrightarrow x + 2$, $g: x \longrightarrow x - 3$, and $h: x \longrightarrow x$ [for] all $x \in R$. Find expressions for
   a) $(f \cdot g)(x)$
   b) $[(f \cdot g)h](x)$
   c) $(fh)(x)$
   d) $(gh)(x)$
   e) $[(fh) \cdot (gh)](x)$

8. In Exercise 7, compare your results for (b) and (e). Th[ey] should be the same. Do you think this result is true fo[r any] three functions f, g, and h, that map real numbers [into] real numbers?

9. Would you say that $f(g \cdot h) = (fg) \cdot (fh)$ for any three func[tions] f, g, and h, that map real numbers into real numbers[?]

10. State which of the following will hold for all functions [f,] g, and h, that map real numbers into real numbers:

$$(f+g)h = fh + gh$$
$$f(g+h) = fg + fh$$

11. Prove that the set of all linear functions is associativ[e under] composition; that is, for any three linear functions f, [g,] and h,

$$f(gh) = (fg)h$$

1-6. Inversion

Quite frequently in science and in everyday life we encounter quantities that bear a kind of reciprocal relationship to each other. With each value of the temperature of the air in an automobile tire, for example, there is associated one and only one value of the pressure of the air against the walls of the tire. Conversely, with each value of the pressure there is associated one and only one value of the temperature. Two more examples, numerical ones, will be found below.

Suppose that f is the function $x \longrightarrow x + 3$ and g is the function $x \longrightarrow x - 3$. Then the effect of f is to increase each number by 3, and the effect of g is to decrease each number by 3. Hence f and g are reciprocally related in the sense that each undoes the effect of the other. If we add 3 to a number and then subtract 3 from the result we get back to the original number. In symbols

$$(gf)(x) = g(f(x)) = g(x + 3) = (x + 3) - 3 = x.$$

Similarly,

$$(fg)(x) = f(g(x)) = f(x - 3) = (x - 3) + 3 = x.$$

As a slightly more complicated example we may take

$$f\colon x \longrightarrow 2x - 3 \text{ and } g\colon x \longrightarrow \frac{x + 3}{2}.$$

Here f says "Take a number, double it, and then subtract 3." To reverse this, we must add three and then divide by 2. This is the effect of the function g. In symbols,

$$(gf)(x) = g(f(x)) = g(2x - 3) = \frac{(2x - 3) + 3}{2} = x.$$

Similarly,

$$(fg)(x) = f(g(x)) = f\left(\frac{x + 3}{2}\right) = 2\frac{x + 3}{2} - 3 = x.$$

In terms of our representation of a function as a machine, the g machine in each of these examples is equivalent to the f machine running backwards; each machine then undoes what the other does, and if we hook up the two machines in tandem, every element that gets through both will come out just the same as it originally went in.

We now generalize these two examples in the following definition of inverse functions.

Definition 1-8. If f and g are functions so related that $(fg)(x) = x$ for every element x in the domain of g ar[...] $(gf)(y) = y$ for every element y in the domain of f, then [...] and g are said to be inverses of each other. In this case [...] f and g are said to have an inverse, and each is said to be [...] inverse of the other.

As a further example of the concept of inverse functions [...] us examine the functions $f\colon x \longrightarrow x^3$ and $g\colon x \longrightarrow \sqrt[3]{x}$. In this [...]

$$(fg)(x) = f(g(x)) = f(\sqrt[3]{x}) = (\sqrt[3]{x})^3 = x$$

and

$$(gf)(x) = g(f(x)) = g(x^3) = \sqrt[3]{x^3} = x$$

for all $x \in R$.

If a function f takes x into y, that is, if $y = f(x$[...] then an inverse g of f must take y right back into x, [...] is, $x = g(y)$. If we make a picture of a function as a mapping [...] with an arrow extending from each element of the domain to its [...] image, as in Figure 1-6a, then to draw a picture of the invers[...] function we need merely reverse the arrows, as in Figure 1-6b.

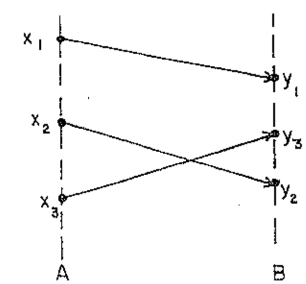


Figure 1-6a. A function.

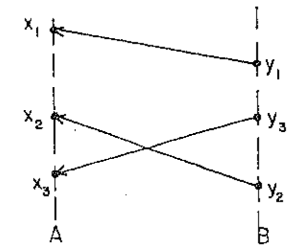


Figure 1-6b. Its inver[...]

We can take any mapping, reverse the arrows in this way, [...] obtain another mapping. The important question for us, at th[...] is this: If the original mapping is a function, will the rev[...] ping necessarily be a function also? In other words, given [...]

function, does there exist another function that precisely reverses the effect of the given function? We shall see that this is not always the case.

The definition of a function (Definition 1-1) requires that to each element of the domain there corresponds exactly one element of the range; it is perfectly all right for several elements of the domain to be mapped onto the same element of the range (the constant function, for example, maps all of its domain onto one element), but if even one element of the domain is mapped onto more than one element of the range, then the mapping just isn't a function. In terms of a picture of a function as a mapping (such as Figures 1-1a and 1-1c), this means that no two arrows may start from the same point, though any number of them may end at the same point. But if two or more arrows go to one point, as in Figure 1-6c, and if we then reverse the arrows, as in Figure 1-6d, we will have two or more arrows starting from that point (as in Figure 1-1b), and the resulting mapping is not a function. Since the word "inverse" is used to describe only a mapping which is a function, we can conclude that not every function has an inverse.

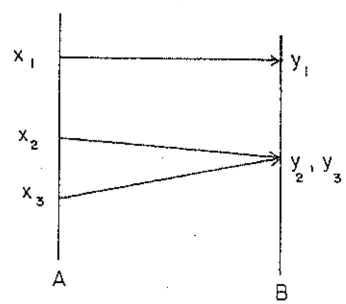


Figure 1-6c.

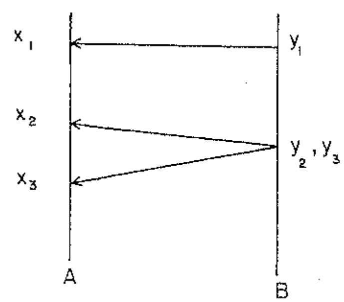


Figure 1-6d.

A specific example is furnished by the constant function f: x—
since f(0) = 3 and f(1) = 3, an inverse of f would have to ma
3 onto both 0 and 1. By definition, no function can do this.

The preceding argument shows us just what kinds of functio
do have inverses. By comparing the situation in Figures 1-6a a
1-6b with the situation in Figures 1-6c and 1-6d, we can see th
a function has an inverse if and only if no two arrows go to th
same point. In more precise language, a function f has an in
verse if and only if $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$. A function
this sort is often called a "one-to-one" function. A formal pr
of this Theorem will be found in Chapter 4.

Exercises 1-6

1. Find an inverse of each of the following functions:
   a) $x \longrightarrow x - 7$ c) $x \longrightarrow 1/x$
   b) $x \longrightarrow 5x + 9$
2. Solve each of the following equations for x in terms of and compare your answers with those of Exercise 1:
   a) $y = x - 7$ c) $y = 1/x$
   b) $y = 5x + 9$
3. Justify the following in terms of composite functions and i
   verse functions: Ask someone to choose a number, but not t
   tell you what it is. "Ask the person who has chosen the num
   to perform in succession the following operations. (i) To
   multiply the number by 5. (ii) To add 6 to the product.
   (iii) To multiply the sum by 4. (iv) To add 9 to the pro
   (v) To multiply the sum by 5. Ask to be told the result o
   the last operation. If from this product 165 is subtracted
   and then the difference is divided by 100, the quotient wil
   the number thought of originally."
   (W. W. Rouse Ball).