Lecture 17: Quadratics June 24, 2010

The standard form of a quadratic is

$$p(x) = a\,x^2 + b\,x + c,$$

where x is a variable, and a, b, and c are constants, with $a \neq 0$.

A quadratic polynomial in factored form is $a(x-\alpha)(x-\beta)$, where a, α and β are constants. This is transformed into standard form by using the distributive law repeatedly:

$$(x-\alpha)\,(x-\beta) = x^2 - (\alpha+\beta)\,x + \alpha\,\beta.$$

An important instance of this is the perfect-square identity:

$$x^2 + 2\,\alpha\,x + \alpha^2 = (x+\alpha)^2. \tag{1}$$

Completing the square

This refers to the following way of rewriting a quadratic, which is motivated by (1):

$$\begin{aligned} x^2 + b\,x + c &= x^2 + 2\,(b/2)\,x + (b/2)^2 - (b/2)^2 + c \\ &= \big[x + (b/2)\big]^2 - \big[(b/2)^2 - c\big]. \end{aligned} \tag{2}$$

We can apply (1) to the case of a quadratic with leading coefficient a:

$$\begin{aligned} a\,x^2 + b\,x + c &= a\left[x^2 + \frac{b}{a}\,x + \frac{c}{a}\right] \\ &= a\left[\left(x + \frac{b}{2a}\right)^2 - \left[\left(\frac{b}{2a}\right)^2 - \frac{c}{a}\right]\right] \\ &= a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4\,a\,c}{4a^2}\right] \end{aligned} \tag{3}$$

Equation (3) shows that

$$a\,x^2 + b\,x + c = 0 \quad \Leftrightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4\,a\,c}}{2\,a}. \tag{4}$$

We can also use equation (3) to select a u-v-coordinate system in which the equation $y = a\,x^2 + b\,x + c$ has a simpler form. Let

$$u = x + \frac{b}{2\,a} \quad , \quad v = y + \frac{b^2 - 4\,a\,c}{4\,a}. \tag{5}$$

Then

$$y = a\,x^2 + b\,x + c \quad \Leftrightarrow \quad v = a\,u^2.$$

The origin of the u-v-coordinate system is called the *vertex* of the graph of $y = a\,x^2 + b\,x + c$. It is at the point where:

$$x = \frac{-b}{2\,a} \quad , \quad y = \frac{-b^2 + 4\,a\,c}{4\,a}. \tag{6}$$

Problem. If we vary one of the coefficients but hold the others fixed, how does the graph of $y = a\,x^2 + b\,x + c$ change? Does the shape of the graph change? How does the vertex move? What happens to the y-intercept?