Debriefing: Exponents

Exponents $1, 2, 3, \ldots$ **abbreviate repeated multiplication.**

If B is a number, then

$$B^n \quad = \quad \overset{\longleftarrow\, n \text{ factors} \,\longrightarrow}{B \cdot B \cdot \cdots \cdot B \cdot B}.$$

Because of this, a *sum at the exponent level* translates into a *product at the base level*:

$$B^{m+n} \quad = \quad \overset{\longleftarrow\, m \text{ factors} \,\longrightarrow}{(B \cdot B \cdot \cdots \cdot B \cdot B)}\overset{\longleftarrow\, n \text{ factors} \,\longrightarrow}{(B \cdot B \cdot \cdots \cdot B \cdot B)} \quad = \quad B^m \cdot B^n$$

If we write exponentiation with base B as a function:

$$g(x) = B^x,$$

then we can state the translation property as follows

$$g(x + y) = g(x) \cdot g(y).$$

This is reminiscent of the property that a function of the form $f(x) = a\,x$ has,

$$f(x + y) = f(x) + f(y),$$

which is a restatement of the distributive law. *But be careful!* With $f(x) = ax$, we have *addition* on the right side. Multiplication by a constant *preserves addition*. Exponentiation with a fixed base *changes addition into multiplication*. We can (in a sense) "distribute the base" over added exponents, but we must change from addition to multiplication.

Note that exponents do *not* distribute over sums:

$$(x + y)^n \neq x^n + y^n.$$

But they *do* distribute over products:

$$(x \cdot y)^n = (x \cdot y)(x \cdot y) \cdots (x \cdot y) = (x\,x \cdots x)((y\,y \cdots y) = x^n\,y^n.$$

Exponents other that $1, 2, 3, \ldots$

They behave the same way, changing addition in the exponent to multiplication at the base level. So, the additive identity—namely 0—in the exponent must become the multiplicative identity at the base level:

$$B^0 = 1.$$

If there is any question about this, then think of it this way:

$$B \cdot B^0 = B^1 \cdot B^0 = B^{1+0} = B^1 = B.$$

If we divide both sides by B,

$$B^0 = B/B = 1.$$

How about negative exponents?

$$B^{-n} \cdot B^n = B^{n-n} = 1.$$

Divide both sides by B^n:

$$B^{-n} = 1/B^n.$$

What happens of we take a power of a power?

$$(B^m)^n \quad = \quad \overset{\longleftarrow\quad n \text{ factors} \quad\longrightarrow}{(B^m)(B^m) \cdots (B^m)(B^m)} \quad = B^{mn}.$$

We can use this to show that there is only one possible meaning we can give fractional powers.

$$B = B^1 = (B^{1/n})^n,$$

so $B^{1/n}$ is the n^{th} root of B.