Lecture 7: And, Or, Not June 10, 2010

So far, we have seen that the language of mathematics starts with *constant symbols* and *variable symbols*. With the inclusion of *functions symbols*, we gain the ability to create *expressions*. An expression summarizes a calculation, which we can exhibit using a tree diagram. At the next level, we find the equality sign, which enables us to make assertions. In addition to the equality sign, the language that we are considering also contains other symbols that can be used like the equality symbol to make assertions. The most common are $\leq$ ("is less than or equal to") and $<$ ("is strictly less than"). When inserted between a pair of expressions, we get an *inequality*. An inequality between arithmetic expressions is true if the numbers denoted by the expressions on either side are in the order suggested by the sign.

Equations and inequalities are called *sentences* if they *do not* contain variable symbols. Sentences are either true or false. An equation or inequality that *does* contain variables is neither true nor false, but becomes true or false when the variables are replaced by constants. In logic, we call such things *sentential functions*. Those combinations of variables that make a sentential function true are called its *solutions*. One of the most common things we do in mathematics is to seek the solutions to an equation or an inequality. The usual methods of finding solutions involve repeated rephrasing, until we reach a sentence whose solutions are obvious.

Example 1. Recall the example from the *Zetetica*: *If two given numbers fall short of a third, and if the ratio of their deficiencies is known, how do we find the third?* Here, we are supposing that we have some indirect information about an unknown number, stated in terms of two known numbers a and b that are less than the unknown as well as the known ratio r of the amounts by which the two fall short of the unknown one. We will translate this into an equation using the variable x to hold the place of the unknown number. The deficiencies are $x - a$ and $x - b$. The ratio is $(x - a)/(x - b)$ (or possibly the reciprocal of that). The assertion about x is:

$$r = \frac{x-a}{x-b} \quad \text{and} \quad a \neq b \quad \text{and} \quad a < x \quad \text{and} \quad b < x.$$

Now, the last three conditions will always apply, so we will not write them. The first condition can be restated:

$$r(x-b) = x-a$$

$$r\,x - r\,b = x - a$$

$$r\,x - x = r\,b - a$$

$$x\,(r-1) = r\,b - a$$

$$x = \frac{r\,b - a}{r-1}$$

Example 2. Solve $24x^2 - 53x + 28 = 0$*.* We multiply both sides by the leading coefficient 24:

$$(24x)^2 - 53(24x) + 24 \cdot 28 = 0.$$

Put a new variable in place of $24x$:

$$u^2 - 53u + 24 \cdot 28 = 0.$$

Now, to factor the left side, we seek numbers whose product is $24 \cdot 28$ and that add to -53. The numbers -21 and -32 work, so:

$$(u - 21)(u - 32) = 0.$$

This $24x = u = 21$ or $24x = u = 32$, so $x = 21/28 = 7/8$ or $x = 32/24 = 4/3$.

Example 3. Solve $5 < |2x + 3|$*.* There are two ways to solve this. The first is based on meaning and the second is based on syntax.

1. This is based on the interpretation of $|x - a|$ as the distance from x to a. The given inequality is equivalent to $5/2 < |x - (-3/2)|$, so it is saying that the distance from x to $-3/2$ is at strictly greater than $5/2$. Thus $1 < x$ or $x < -4$.
2. The definition of $|E|$ is:

   $$|E| = \begin{cases} E, & \text{if } 0 \leq E; \\ -E, & \text{if } 0 > E. \end{cases}$$

   Therefore,

   $$\begin{aligned} 5 < |2x + 3| &\Leftrightarrow 0 \leq 2x + 3 \text{ and } 5 < 2x + 3 \text{ OR } 0 > 2x + 3 \text{ and } 5 < -(2x + 3) \\ &\Leftrightarrow 5 < 2x + 3 \text{ OR } -5 > (2x + 3) \\ &\Leftrightarrow 1 < x \text{ OR } -4 > x \end{aligned}$$

Observe that in examples 2 and 3, we translated equations (or inequalities) into complex statements that involved the *logical connectives* “and”, “or” and ”not”.

Problems.

1. Solve $5 < \big||x| - 10\big|$.
2. Find an inequality (with variable x) whose solution set is a union of 4 disjoint closed intervals.
3. Solve $0 < \big||x - y| - x\big|$.
4. Solve $0 < \big||ax + by + c| - dx\big|$.