What is a conversion factor? June 13, 2011. Revised, June 18, 2011.

Suppose there are two units of measurement at hand—inches and centimeters, say. Suppose we know the measure of an object with respect to one unit and we wish to compute the measure with respect to the other. We can achieve this by multiplying by the an appropriate *conversion factor*.

For example, there are (by international agreement) exactly 2.54 centimeters in one inch, and therefore, the conversion factor that changes a number of inches to the number of centimeters that spans the same length as the inches is 2.54. If a piece of wire is carefully cut to exactly 69.5 inches, then its length in centimeters is $2.54 \times 69.5 = 176.53$.

It is typical to think of the 2.54 as a "rate," and to give it the label "centimeters per inch." If an object is measured in inches—producing the number n—and is also measured in centimeters—producing the number m—then $m/n = 2.54$. It does not matter how big or small the objects are. If we measure in both ways over and over, then as long as the measurements are accurate, we always get 2.54.

Sources of confusion

When working with conversion factors, it is easy to get confused. The reason seems to be related to the difficulty that some people—myself included—have in translating certain kinds of statements into equations. For example:

1) *Plato's Trail Mix has 10 raisins for every 3 nuts. In a certain bag of Plato's Trail Mix, the number of raisins is R and the number of nuts is N. Which is correct?*

   a) $10R = 3N$

   b) $3R = 10N$

2) *On June 18, 2011, the currency exchange rate was 1 euro = 1.4315 US dollars. If I had D dollars on that day, then what equation should I use to find out the number E of euros that my fortune was then worth?*

   a) $D = 1.4315\,E$

   b) $E = 1.4315\,D$

There are several possible sources of confusion:

1) If we pair the words with the most-closely-related symbols, then the *order of the words* in the statement, "Ten raisins for every three nuts," is the same as the *order of the symbols* in the *incorrect* equation, "$10R = 3N$."
2) One might misunderstand the variable symbols, taking them as *names for the objects themselves* rather than as *names for the numbers that count or measure the objects*. When this happens, the equals sign is not seen as stating a relationship between numbers—which is the *only* job ever has in algebra—but as a relationship between situations. "$10R = 3N$" is viewed as shorthand for, "Getting ten raisins in a serving is the same as getting 3 nuts."

3) The variable symbols E and D might be misinterpreted as references to the *worth of a single unit*, rather than to *a number of units*. We might measure the worth of a thing by the number of grams of bread for which it may be exchanged. If we write e for the value of a euro and d for the value of a dollar, then $e = 1.4315\, d$. But the E and D do not stand for the worth of the euro and the dollar, they stand for the numbers of euros and dollars that must enter a fair exchange, and these are related by the equation $D = 1.4315\, E$.

These confusions may creep into anyone's thinking. A 14-year-old lives on in everyone's mind, and the trick is not to silence him/her, but just to slow him/her down and demand that he/she be careful. *"Stop, check and be sure!"* is a mathematical habit of mind that saves us, and it is one of the most valuable things that we can share with those whom we teach.

Measure numbers and conversion factors

Each act of measuring involves: 1) the thing measured, 2) the unit by which it is measured and 3) the number that comes out in the end. That last thing is called the "measure number." The act of measuring involves something like division, but it begins with things rather than with numbers. When we measure the length of a piece of rope, we find out how many times the measuring stick fits end-to-end along the rope, and that number is the measure number (*of* the rope *in* measuring-stick units).

In ancient times, this was called a ratio. The measure of the rope is the *ratio* of the rope to the measuring stick. Today, we come across many different opinions about what a ratio "really" is. It does not matter what we call it, but we must bear in mind that measurement is a passage from things to numbers. It is a process where things go in and numbers come out.[1]

In order for anyone to know the meaning of the measure number that comes out of a measurement process, one needs to know what went in. Therefore, the name of the unit is written next to the measure number. It records the unit used. Mathematicians have shown that such "unitized" or "dimensioned" numbers may be treated as special kinds of mathematical objects, similar to pure numbers, but distinct from them. This does not mean, however, that we *must* view them that way. A different approach is to treat measure symbols, such as "inches" or "pounds," as commentary that is supplied in order to link a measure number back to essential information about the measurement process from which it arose. This is the approach that I prefer.

A conversion factor is simply the measure number that arises when one unit is measured by another; it is the ratio of one unit to another. Since conversion factors are always used in a context where two different units come into play, conversion factors include three

[1] Isaac Newton suggested that this is exactly what numbers are—the things that result when we measure. In his *Universal Arithmetick*, he wrote, "By Number we understand not so much a multitude of Unities as the abstracted Ratio of any Quantity to another Quantity of the same Kind, which we take for Unity."

pieces of information: a measure number, the unit doing the measuring , and the unit being measured. For example, the conversion factor "2.54 centimeters per inch" contains the information that when an inch is measured using centimeters as a unit, the measure number that results is 2.54. When I report the measure of an ordinary thing—e.g., when I say that my pencil measures 18 centimeters—I report the measure number and the unit in the customary manner. When I tell you a conversion factor, I report the measure number and the unit in the same manner, but I also include the phrase "per inch." If I were planning to use my pencil to measure things, then I might want to be able to convert from pencils to centimeters, so I would report a conversion factor of "18 centimeters per pencil."

Multiplying ratios

The ratio of A to B times the ratio of B to C is equal to the ratio of A to C:

$$\frac{A}{B} \cdot \frac{B}{C} = \frac{A}{C}.$$

This is clearly true if A, B and C are numbers. It is also true if A, B and C are things with the same measurable attribute, and the symbol A/B is the measure of the top by the bottom:

$$\text{(pencil measured by inch)} \cdot \text{(inch measured by cm.)} = \text{pencil measured by cm.}$$

If measure numbers with their units are written, then this appears

$$\text{(7.09 inches)} \cdot \text{(2.54 cm. per inch)} = \text{(18.0 cm.)}$$

The conversion factor 2.54 is not the ratio of a centimeter to an inch, even though its "dimension" is cm./inch. It is the ratio of an inch to a centimeter, or the "value of an inch measured in centimeters."

Changing coordinates

Much of this is relevant to a change of coordinates by a change of scale. The x-coordinate of a point on a line is the signed measure of distance of that point from the origin of the x-system, using the unit of the x-system as a standard. If the u-system has the same origin, the equation that relates the x-system and the u-system involves a conversion factor:

$$u(P) = c \cdot x(P).$$

We can think of the conversion factor here as given in "u-units per x-unit," i.e., as a measure of the x-unit by the u-unit. Thus, the u-unit is smaller than the x-unit if and only if the absolute value of the conversion factor is larger than 1.

For example, if we compare the graph of $y = \sin x$ to the graph of $y = \sin 3x$, then the latter can be viewed as the graph of $\sin u$, where the u-system has a smaller unit than the x-system. Since the u-unit is smaller, the graph oscillates more rapidly.