**Lecture 9: Coordinates on the plane, distance formula, equations and their graphs**
June 14, 2010

I once asked a class of undergraduates why the graph of a linear equation is a line. One student looked at me, kind of tilted her head, and said, "Well, duh! Why do you think it's called 'linear'?" Well, it was a trick question. The graph of a linear equation is not always a line. I can show you. Here (in red) is a portion of the graph of $r = \frac{3}{4}\theta - \frac{5}{2}$:
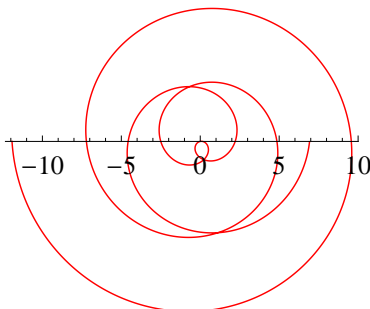
Of course, r and θ are polar coordinates, but I mean to make a point. The reason that the graph of a linear equation is a line must have something to do with the special nature of the *cartesian coordinate system*, because in other coordinate systems, this is not true. So, the real question is, "What is it about the cartesian coordinate system that assures that the graph of a linear equation is a straight line?"

STANDARD COORDINATE SYSTEM IN A PLANE
Présenté à la manière d'Irma Rombauer, auteure de *The Joy of Cooking*.

Take:
> **1 unmarked plane**
> **1 unit of length**

In the plane, select:
> **2 lines that are perpendicular to one another**
> **a direction on each of the lines**

Using the unit of length, the point where the lines cross as the origin and the selected directions create a coordinate system on each of the lines following the recipe for Coordinate System on a Line (in previous lecture). Call one of the lines the x-axis and call the other the y-axis, and call the coordinate functions on them x and y respectively. Now extend the x and y functions to the whole plane as follows. Given a point P in the plane, draw a line through P perpendicular to the x-axis. Define $x(P)$ to be the x-coordinate of the point where that line crosses the x-axis. Define $y(P)$ similarly. Serve in fifth grade.

What does the Common Core say?

Paragraph 5.G.1 says that students in the fifth grade ought to: *Use a pair of perpendicular number lines, called axes, to define a coordinate system, with the intersection of the lines (the origin) arranged to coincide with the 0 on each line and a given point in the plane located by using an ordered pair of numbers, called its coordinates. Understand that the first number indicates how far to travel from the origin in the direction of one axis, and the second number indicates how far to travel in the direction of the second axis, with the convention that the names of the two axes and the coordinates correspond (e.g., x-axis and x-coordinate, y-axis and y-coordinate).*

The distance formula

In the picture below, we have drawn two points (P and Q) as well as the lines needed to find their x- and y-coordinates. Notice that P and Q are opposite vertices of a rectangle with width $|x(Q) - x(P)|$ and height $|y(P) - y(Q)|$.
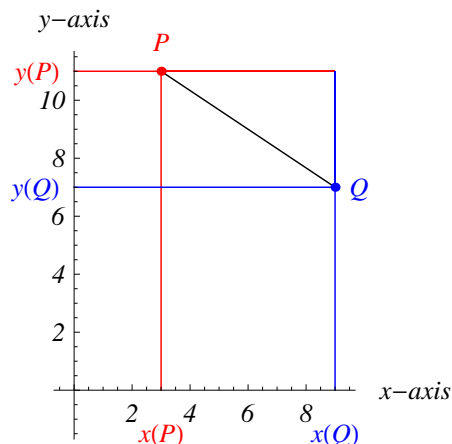
The distance between the points is denoted $\text{dist}(P, Q)$. It is the length of the diagonal of the rectangle. By the Pythagorean theorem,

$$\text{dist}(P, Q) = \sqrt{\big(x(Q) - x(P)\big)^2 + \big(y(Q) - y(P)\big)^2}$$

Equations in two variables and their graphs

Suppose $\Theta(x, y)$ is any equation in which x and y are the only variables. For example $\Theta(x, y)$ might be $x^2 - y^2 = 16$. We say that a specific pair of numbers (a, b) is a solution of $\Theta(x, y)$ if $\Theta(a, b)$ is true. For example, $x^2 - y^2 = 16$ has $(5, 3)$ as a solution, because $5^2 - 3^2 = 16$. The set of all solutions of $\Theta(x, y)$ is called the *graph* of $\Theta(x, y)$.

When introducing the concept to students, it may be helpful to say that the equation $\Theta(x, y)$ as a "point-tester". If we plug the coordinates of a point into the equation, the result may or may not be true. If it is true, the point passes the test, and the set of all points that pass form the graph of the equation.

In most cases, it is very difficult to figure by hand out what the graph of an equation looks like (though we can easily produce pictures on computers). The graph of $(x^2-4)(x-1)^2+(y^3-2y+1)^2 = 0$ is shown below:

2
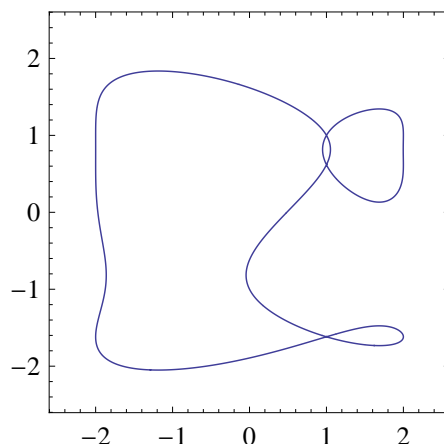
The circle

In a few special cases the graph of an equation is easy to understand and picture. For example, a point (x, y) is on the graph of the following equation if and only if it is at distance 10 from $(3, 2)$:

$$10 = \sqrt{(x-3)^2 + (y-2)^2}.$$

In other words, the graph is a circle. In general if $r > 0$ and a, b are numbers, then the graph of

$$r = \sqrt{(x-a)^2 + (y-b)^2}$$

is a circle of radius r centered at (a, b). This equation may be rewritten in many equivalent forms, e.g.:

$$r^2 = (x-a)^2 + (y-b)^2$$

or

$$r^2 = x^2 + y^2 - 2ax - 2by + a^2 + b^2$$

Lines and linear equations

If we want to make a connection between lines and the cartesian coordinate system, then we need a description of lines in a language that a coordinate system can talk about. The distance formula enables a coordinate system to talk about distance, so we ought to look for a description of lines in terms of distance. We have such a description:

Problem 1. Suppose P and Q are two points. Show that

$$\{\, Z \mid Z \text{ is a point in the plane and } \mathrm{dist}(P, Z) = \mathrm{dist}(Q, Z) \,\}$$

is a line. (In other words, show that the set of points equidistant from P and Q is a line.) Show also that if ℓ is any line, then there are points P and Q so that ℓ is the set of points equidistant from P and Q.

Problem 2. Suppose that P has coordinates (a, b) and Q has coordinates (c, d). Let Z be a point with coordinates (x, y). Then Z is equidistant from P and Q if and only if:

$$\sqrt{(x-a)^2 + (y-b)^2} = \sqrt{(x-c)^2 + (y-d)^2}. \tag{$*$}$$

Show that $(*)$ simplifies to a linear equation of the form $C = Ax + By$, where $A = 2(a - c)$, $B = 2(b - d)$ and $C := (a - c)(a + c) + (b - d)(b + d)$. Conclude that the line of points equidistant from P and Q is the solution set of a linear equation.

Problem 3. Let A, B, C be any three numbers with at least one of A and B not equal to zero. Show that there are numbers a, b, c and d such that $A = 2(a - c)$, $B = 2(b - d)$ and $C := (a - c)(a + c) + (b - d)(b + d)$. Conclude that the graph of $Ax + By = C$ is the set of points equidistant from the point with coordinates (a, b) and the point with coordinates (c, d).