What geometry should high school teachers know?
What should they teach?

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On this question there is no shortage of opinion, variously motivated and held with varying strength of conviction. Ultimately, these questions have no answer that is *not* opinion. But opinions have their value, and among opinions some certainly deserve our attention (while some others, perhaps, do not). The opinions of the LSU mathematics faculty ought to interest us. There are a number of reasons why—not the least of which is the fact that high-school students, if they so aspire, become college students.

At the end of last semester, I asked each of my colleagues to suggest two geometry problems. The first was to be a problem that a good high-school teacher ought to be able to discuss intelligently. The second was to be representative of the kind of geometry problem that an in-coming freshperson ought to be able to deal with. I hoped that the answers would provide a sense of the collective opinion of the math department. It would be raw and ragged about the edges—not a polished syllabus, but instead something that might show you what could happen in an informal conversation with a professor who was probing to discover what you knew.

If in this course you can become comfortable with questions similar to those below, then you can be confident in the company of mathematicians. You may even think of yourself as belonging to their company. And you will be prepared to send your students into their company.

Of my 42 colleagues, 13 responded. Here are the problems they suggested:

1.a. A high school teacher who understood concepts related to "inversion in a circle" would impress me. Suppose that A is the center of a circle γ, B is outside γ and C is a point on γ such that BC is tangent to γ. Let D be the foot on AB of the altitude of triangle ABC from C. Then B and D are defined to be inverses of one another with respect to γ. A well-informed teacher might know that if circles γ and λ are orthogonal, then the set of inverses with respect to γ of points on λ is λ itself. Or (s)he would be able to make perceptive comments on the following: fix two points A and B and consider the family $\mathcal{A}$ of all the circles through those points. Find a circle that is orthogonal to every circle in $\mathcal{A}$. Find *all* the circles orthogonal to every circle in $\mathcal{A}$.

b. Let γ be a circle with center A. Let B, C and D be points on the circle. A good high school student would know the relationship between the measure of angle BAC and the measure of angle BDC and would be able to explain it.

2.a. A good high school teacher should be aware of the ways in which the notion of distance has become generalized and abstracted; the idea of a general metric space, the idea of a norm in a linear space, the various ways of measuring distance between functions, the idea of length of a path in Riemannian geometry, *etc.*

b. High school students should know all about similar triangles, the law of cosines, *etc.*

3.a. For teachers: Consider a non-Euclidean geometry in which, through a point not on a line, there is always more than one line parallel to the given line. Let ABC be a triangle with a right angle at point A. Show that there is a number $k < \frac{\pi}{2}$ so that the measure of angle ABT is less than k for all points T on line AC.

b. For students: Let triangle ABC be inscribed in a circle, with AB a diameter. Show that angle C is a right angle. (This is sometimes known as Thales' Theorem.)

4.a. For teachers: A book of mathematical puzzles contained the following problem: *Find the dimensions of a cube whose surface area is numerically equal to its volume.* The answer given was that the cube is 6 inches on a side. How many things can you find wrong with this?

b. For students: You know the formulas for the area of a circle (πr^2, where r is the radius) and the volume of a rectangular solid (length $\times$ width $\times$ height). You may or may not know the formula for the volume of a sphere of radius r, but in any case you are not allowed to use this in your answer. You ask two fellow students if they know this formula: Albert tells you that it's $\frac{4}{3}\pi r^2$, and Betty tells you that it's $5r^3$. They're both wrong; your job is to pick *one* of these answers and explain why it's wrong.

5.a. Any competent high school geometry teacher should have knowledge of Non-Euclidean Geometries. In particular, they should understand what a system of axioms is and why and how axiom systems are used. They should also have a general familiarity with the history of geometry.

b. Any high school student coming to LSU should have heard about the impossibility of certain geometric constructions "by straightedge and compass," such as trisecting an arbitrary angle, doubling the cube, or squaring the circle.

6.a. For high school teachers, I suggest the following: Approximation of π by sub- and super-inscribing regular polyhedra around a circle, and dissecting them into isosceles triangles.

b. Everyone ought to be able to think about the map $t \mapsto (\cos t, \sin t)$ as the "wrapping" function—the real line is wound around the unit circle.

c. Another topic that should be covered but is apparently not—I did not see it until I was in graduate school—is the "golden mean" and its properties. Looking at classical architecture would be a good tie-in.

7.a. What does it mean for two triangles to be similar? What does it mean for two triangles to be congruent? What criteria serve to show that two triangles are congruent?

b. Prove the Pythagorean Theorem (without using the Law of Cosines).

8.a. Consider the set of all n-gons inscribed inside a given circle. Show that the n-gons with the maximum area are precisely the regular ones.

b. For incoming freshmen: Is it possible to walk one mile north, then one mile east, and finally one mile south and end up in your starting position? If so, describe all such routes.

9.a. Problem for teachers: *Part 1*: Erect a perpendicular to a given line (not necessarily through a given point) WITHOUT using a compass, but rather using only a straightedge and a "scale," i.e., an instrument with which to lay off a single, fixed segment on a given line from any given point on that line. (Hint: The three altitudes of any triangle meet in a single point. More precisely, the CONVERSE of this theorem is also true.) *Part*

2: Use the above construction to construct the square root of $a^2 + b^2$, for any given segments of lengths a and b.

b. Problem for students: *Part 1*: Multiply any two given positive real numbers using ruler and compass. *Part 2*: Construct the square root of any given positive real number, using ruler and compass.

Some contributors just suggested a number of problems for teachers and students.

10.a. You have an unmarked circular disk of wood, which you are going to use to make the seat of a milking stool. The problem is to determine where to attach the 3 legs. They must be placed symmetrically around the center, but the center itself is not marked. To solve this problem, you can use only a straightedge, a piece of string, a pencil and a thumbtack.

b. Discuss the shuffling squares puzzle. What strategy do you use to get the 15 squares back in order?

c. A well-known riddle goes as follows: A man travels south 1000 miles, then travels east 100 miles, sees a bear, then travels north 1000 miles and arrives at the point where he stated. In some versions, he shoots the bear, but we will let the bear live. In any case, the problem is, "What color was the bear?" The answer is "White!" because there is only one place on earth where this sequence of events could occur—the man must have started at the North Pole. However, there are infinitely many points on the surface of the earth from which a man can begin a trip, travel south 1000 miles, then travel east 100 miles, then travel north 1000 miles and arrive at the point from which he stated. (There just aren't any bears within viewing distance of the southern parts of these trips.) What are all the points from which such trips may begin?

d. The teacher said, "Cut a square into two pieces that are identical in shape." The student said, "I cannot do it, because I don't know what to do with th center point." What *does* happen to the center point?

11.a. Prove that in a 30-degree–60-degree right triangle, the hypotenuse is twice the side opposite the 30-degree angle.

b. Prove that corresponding angles are equal (when a transversal intersects two parallel lines).

c. Prove that alternate interior angles are equal (ditto).

d. Prove that an exterior angle of a triangle is equal to the sum of the two opposite interior angles.

e. Prove that if A and B are opposite ends of a diameter of a circle with center O, and P is a point of the circle distinct from A and B, then, the angle BOP is twice the angle OAP.

12.a. *Proposition*. Base Angles of an isosceles triangle ABC are congruent. *Proof*. Assume $AB = AC$. Then $BAC \cong CAB$ by SAS. Thus the corresponding angles B and C are congruent.

b. Prove the Pythagorean Theorem by cutting out 4 copies of triangle ABC, and one cutout each of the squares on side a, b, and c. Then play with the 'jigsaw puzzle.'

Later, this contributor wrote:

I should have cited something like the Plancherel theorem as a modern version of the Pythagorean theorem, and Holder-Minkowski as the triangle inequality, another vital thing for students to know from geometry. And here is another which I think is very important: Students should have studied at least one of the limit arguments of the Greek geometers—say the proof that the areas of two circles are in the same proportion as the squares on their diameters—*e.g.*, by the 'principle of exhaustion.' It would be nice to know why conic sections are sections of cones, too—at least for astronomers, physicists, nuclear engineers, *etc*. But maybe that is a luxury of 'an expensive classical education!'

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13. *One contributor wrote wrote*:

Here are my thoughts about 4005 and geometry. I think that the book *Roads to Geometry* most closely corresponds to anything I have seen of my vision of what should be done in the course. Unfortunately, I am never able to cover this amount of material. From that book, here is what I would like to cover, chapter by chapter:

1. I want students to develop some familiarity of working with abstract geometric axiom systems. Finite geometries are rather nice for this.
2. (omit 2.4 and 2.5) The authors state in the Preface: "Through the choice of the SMSG postulate set as a basis for the development of plane geometry, this book avoids the pitfalls of many 'foundations of geometry' texts which encumber the reader with such a detailed development of preliminary results that many other substantive and elegant results are inaccessible in a one semester course." Amen.)
3. (sections 3.1-3.4)
4. (sections 1-6 and section 9) In regard to 4.9, I think that some version of constructions should be touched on, although now I am more inclined to do these in the context of "Geometer's Sketchpad."
5. (sections 1-3) I think sections 5.2 and 5.3 are very high priority items for this course. Students should definitely be exposed to ideas of coordinate or analytic geometry and also the basics of rigid motions.
6. First pick up sections 3.5 and 3.6, then section 6.3 and highlights from 6.6) (this material is more optional in nature—if hard pressed for time, one could limit exposure to non-euclidean geometry to be that in section 2.7 together with the neutral geometry of Chapter 3.

To the extent possible, I think it is quite good pedagogy to reinforce as many ideas as possible with assignments on Geometer's Sketchpad. There are some nice worksheets available in this regard.

What high school students coming to college should know about geometry:

- basic area and volume formulas;
- basic analytic geometry of the plane (distance formula, slope, how to determine if lines are parallel and perpendicular, midpoints, equations of circles, etc.);
- basic notions of similarity and proportion for triangles;

- very basic facts of the euclidean geometry of triangles, right triangles, parallelograms, rectangles, squares, and circles.