MATH SATURDAY
at the LSU Highland Road Observatory

May 10, 2003

LATE BREAKING NEWS:

A teacher has been arrested in the United Kingdom in possession of compasses, protractors, and a straight edge. It is claimed he is a member of the Al Gebra movement bearing weapons of math instruction.

We now return to scheduled programming:

"From Long Division to Number Theory"

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1. Every fraction has a decimal expansion that you can compute by performing a long division. For example,

$$1/7 = 0.142857142857142857 14\ldots$$

Here's some room for you to actually try the division. I'll be talking about this later, so it's really worth taking the time to activate your "long division neurons".

2. If you look carefully, you see that the digits 1, 4, 2, 8, 5 and 7 keep repeating in that order. We can take advantage of this to abbreviate the decimal expansion of 1/7 as

$$0.\overline{142857}.$$

In fact, the decimal exansion for every **rational number** (*i.e.*, ratio of whole numbers, such as 1/7, 27/55 or 200/13) eventually falls into a cycle, though it may not begin right away:

$$121/148 = 0.81756756\ldots = 0.81\overline{756}$$

What's the guarantee? How can we prove that the decimal expansion of every rational number eventually falls into a repeating cycle of digits? Here's room to think:

Before we go on, what's 2/7 *in decimal notation?*

3. If we look at the decimal expansions for some carefully selected fractions, we see some amazing patterns. For example:

$1/7 = 0.\overline{142857}$	$1/13 = 0.\overline{076923}$	$7/13 = 0.\overline{538461}$
$2/7 = 0.\overline{285714}$	$2/13 = 0.\overline{153846}$	$8/13 = 0.\overline{615384}$
$3/7 = 0.\overline{428571}$	$3/13 = 0.\overline{230769}$	$9/13 = 0.\overline{692307}$
$4/7 = 0.\overline{571428}$	$4/13 = 0.\overline{307692}$	$10/13 = 0.\overline{769230}$
$5/7 = 0.\overline{714285}$	$5/13 = 0.\overline{384615}$	$11/13 = 0.\overline{846153}$
$6/7 = 0.\overline{857142}$	$6/13 = 0.\overline{461538}$	$12/13 = 0.\overline{923076}$

$1/41 = 0.\overline{02439}$	$11/41 = 0.\overline{26829}$	$21/41 = 0.\overline{51219}$	$31/41 = 0.\overline{75609}$
$2/41 = 0.\overline{04878}$	$12/41 = 0.\overline{29268}$	$22/41 = 0.\overline{53658}$	$32/41 = 0.\overline{78048}$
$3/41 = 0.\overline{07317}$	$13/41 = 0.\overline{31707}$	$23/41 = 0.\overline{56097}$	$33/41 = 0.\overline{80487}$
$4/41 = 0.\overline{09756}$	$14/41 = 0.\overline{34146}$	$24/41 = 0.\overline{58536}$	$34/41 = 0.\overline{82926}$
$5/41 = 0.\overline{12195}$	$15/41 = 0.\overline{36585}$	$25/41 = 0.\overline{60975}$	$35/41 = 0.\overline{85365}$
$6/41 = 0.\overline{14634}$	$16/41 = 0.\overline{39024}$	$26/41 = 0.\overline{63414}$	$36/41 = 0.\overline{87804}$
$7/41 = 0.\overline{17073}$	$17/41 = 0.\overline{41463}$	$27/41 = 0.\overline{65853}$	$37/41 = 0.\overline{90243}$
$8/41 = 0.\overline{19512}$	$18/41 = 0.\overline{43902}$	$28/41 = 0.\overline{68292}$	$38/41 = 0.\overline{92682}$
$9/41 = 0.\overline{21951}$	$19/41 = 0.\overline{46341}$	$29/41 = 0.\overline{70731}$	$39/41 = 0.\overline{95121}$
$10/41 = 0.\overline{24390}$	$20/41 = 0.\overline{48780}$	$30/41 = 0.\overline{73170}$	$40/41 = 0.\overline{97560}$

Hey! I didn't say the patterns were going to clobber you over the head! Take some time. Look carefully. What do you see?

Can you explain where these patterns are coming from?

4. Let's look at 1/7. The decimal digits are determined by the remainders that we get as we move through the long division. After finding a remiander, we bring down a 0. When we do this, we are in effect multiplying the remainder by 10. We then find out how many time 7 goes into that, and write that number as the next decimal digit. So each digit depends entirely on the reminder arrived at just before it.

Let's look at the process in more detail. We may view the 1 that appears inside the division figure as the first remainder. We multiply this by 10 (getting 10) and then we find that 7 goes into 10 1 time—giving us our first decimal digit. When dividing 7 into 10, we got a remainder of 3, so now we work with this 3. We multiply the 3 by 10 to get 30, divide 7 into 30. It goes 4 times—this is our second decimal digit—and get a remainder of 2 to continue with. And so it goes.

Now can you explain why the repeating unit in each of the multiples of 1/7 is simply a shifted version of the repeating unit in 1/7? But what's going on with 1/13? and 1/41?

5. When we do arithmetic operations and then take reminders after division by 7, we are doing "arithmetic mod 7". An important fact about doing arithmetic mod 7 is that you can take reminders at any time in the process. It won't affect the outcome. To see what I mean, suppose we multiply 10 by 9 to get 90. The remainder when you divide this 90 by 7 is . . . *Ta-daaa*: 6. But you could have figured remainders first. The remainder of 10 is 3 and the remainer of 9 is 2. We could have gotten our 6 by multiplying these numbers.

How about multiplying 12 and 13 mod 7?

Why would it be reasonable for me to say: "In mod 7, there's no difference between 10 and 3."? Why is this similar in spirit to saying: "In regular arithmetic, there's no diffence between $\frac{5}{10}$ *and* $\frac{1}{2}$*."?*

In mod 7, what are the powers of 3? (Answer: 3, 2, 6, 4, 5, 1, 3, . . .)

Where have you encountered the sequence "3, 2, 6, 4, 5, 1, 3, . . ." before?

6. Now, let's look at the 13^{th}s.

What are the powers of 10 mod 13?

How does your answer to the previous question help explain why the repeating unit in the decimal expansion of $1/13$ *has length 6?*

Why is the repeating unit of $2/13$ *NOT a shifted version of the repeating unit of* $1/13$*?*

7. *And the* 41^{st}*s?*

8. *What's the following got to do with what went before?*

$$\begin{aligned}
9 &= 3^2 \\
99 &= 3^2 \cdot 11 \\
999 &= 3^3 \cdot 37 \\
9999 &= 3^2 \cdot 11 \cdot 101 \\
99999 &= 3^2 \cdot 41 \cdot 271 \\
999999 &= 3^3 \cdot 7 \cdot 11 \cdot 13 \cdot 37 \\
9999999 &= 3^2 \cdot 239 \cdot 4649 \\
99999999 &= 3^2 \cdot 11 \cdot 73 \cdot 101 \cdot 137 \\
999999999 &= 3^4 \cdot 37 \cdot 333667 \\
9999999999 &= 3^2 \cdot 11 \cdot 41 \cdot 271 \cdot 9091 \\
99999999999 &= 3^2 \cdot 21649 \cdot 513239
\end{aligned}$$