The Poisson Distribution

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The Poisson distribution $P(k, a)$ tells you the probability of getting k chocolate chips in a cookie if the average number of chips per cookie is a. You are to assume that the cookie has been cut out from a huge batch of cookie dough in which the chocolate chips are randomly distributed. How huge? Actually, we want to consider an infinite amount of dough. As we will see, bigger and bigger batches give results that are closer and closer to the results that could be expected from an infinite batch.

We shall find a formula for the Poisson distribution. To do so, let us measure the amount of dough in units of weight such that one cookie weighs one unit. Say the amount of dough is d—that is, we have enough dough to make d cookies. The number of chips in the dough is ad.

Pick one chip to think about. Grab out enough dough for one cookie. The probability of that chip being in that cookie is $1/d$, since we could make d cookies from the whole batch, and the chip we picked to think about has an equal chance of being in any one of them.

When we grab the dough, we may view each chip as being on trial. A chip wins if it gets into the cookie, and it looses if it's left behind. Since all the chips are on trial, we have ad trials. The number of successes is the number of chips in our cookie.

The binomial distribution

$$B(n, k, p) := \binom{n}{k} p^k (1-p)^{n-k}$$

tells us the probability of k successes in n trials, where the probability of success in any one trial is p. Using this, we see that the probability of having k chips in a cookie is

$$B(ad, k, 1/d) := \binom{ad}{k} (1/d)^k (1-(1/d))^{ad-k}.$$

We want to determine what happens to this quantity as d gets larger and larger. Note that

$$\binom{ad}{k} (1/d)^k = \frac{ad}{d} \frac{ad-1}{d} \cdots \frac{ad-k+1}{d} \frac{1}{k!}.$$

As d gets bigger and bigger, this number approaches

$$\frac{a^k}{k!}.$$

Also, note that

$$(1-(1/d))^{ad-k} = (1-(1/d))^{-k} \big((1-(1/d))^d\big)^a.$$

As d gets bigger and bigger, $(1-(1/d))^{-k}$ gets closer and closer to 1. Also—and this is a fact from calculus—as d gets bigger and bigger, $(1-(1/d))^{d}$ gets closer and closer to $1/e$. Hence, we see that when d is very large,

$$B(ad, k, 1/d) \cong \frac{a^k}{k!}e^{-a}.$$

Therefore

$$P(k, a) = \frac{a^k}{k!}e^{-a}. \tag{$*$}$$

Note that the sum of the probabilities of all possible outcomes must be 1. In other words:

$$1 = P(k, 0) + P(k, 1) + P(k, 2) + \cdots.$$

Using $(*)$, we see:

$$\begin{aligned} 1 &= e^{-a} + ae^{-a} + \frac{a^2}{2!}e^{-a} + \frac{a^3}{3!}e^{-a} + \cdots \\ &= e^{-a}\left(1 + a + \frac{a^2}{2!} + \frac{a^3}{3!} + \cdots\right), \end{aligned}$$

yielding the familiar formula:

$$e^a = 1 + a + \frac{1}{2!}a^2 + \frac{1}{3!}a^3 + \cdots.$$