Two Lesson Packages on Connections Between the Geometry and Algebra of Rigid Motions in the Plane

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Introduction

This report is a record of activities performed in accordance with the directives of the NSF project "Mathematics for Future Secondary Teachers. The purpose of the project is to formulate a set of units of lesson segments which are appropriate for pre-service secondary teachers. Our part in the project was to, as a team, develop and report on a lesson presented in the academic year 2001-02.

Our team included a math educator, mathematician, two undergraduate students, two in-service teachers who took the course, and one teacher. The basic concepts of the lesson were piloted in a special topics course (Introduction to *Mathematica*™), linear algebra, and college geometry. The special topics course project occurred in the fall semester, and the linear algebra and college geometry lessons occurred in the spring semester. The spring and fall presentations were different in both duration and goals.

This report will conform in structure to the report *Toward a Set of Lessons on Accuracy and Uncertainty in Measurement for Future High-School Teachers*, by James J. Madden, a Principal investigator of the study.

I. Mathematical Discussion

The lesson was the mathematical representation of objects in the plane and transformational geometry using a homogeneous coordinate system and 3x3 matrix representations of motions in the plane. Homogeneous matrices and coordinates are used in computer graphics to represent motions of both two and three dimensional objects. This lesson deals exclusively with planar objects.

While a lesson on homogeneous matrix representation of rigid motion is contained in the college geometry course text[1], it does not appear in standard high school algebra and geometry texts. The typical discussion of transformation matrices in the high school text involves 2x2 matrix multiplication, which allows for only a very limited set of isometries in the plane. The purpose of this lesson was to give teachers a simple introduction to the concept for the plane, with the following expectations for use:

1) as an authentic application of matrix multiplication;
2) as an extension of the textbook application of matrix multiplication to transformational geometry;
3) as a connection between composition of linear functions and matrix multiplication.
4) as an application of the connection between the slope of a line and the tangent of the angle of inclination.

Each of these applications involves multiple representations of rigid motion in the form of a geometric construction using patty paper, a reflecting tool, or dynamical geometry software, as well as an algebraic representation using homogeneous coordinates and 3x1 matrix representations of points in

[1] See Meyer, Walter. (2000). Geometry and Its Applications, Chapter 6.

the plane. The concept is rich in connections; it can be the organizing concept of a project or limited to a small unit.

The homogeneous representation of the point (x, y) in the coordinate plane is the ordered triple (x, y, 1). The following are homogeneous matrix representations of some of the basic motions of the plane.

1) The matrix of rotation centered at the origin (0, 0) is $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$, where θ is the directed angle of rotation.

2) The matrix of translation by the vector (h, k) is $\begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix}$.

3) The matrix of reflection in the line Ax + By = 0 is $\begin{bmatrix} \frac{B^2-A^2}{A^2+B^2} & \frac{-2AB}{A^2+B^2} & 0 \\ \frac{-2AB}{A^2+B^2} & \frac{A^2-B^2}{A^2+B^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$, A $\geq$0 .

For example, we can rotate the point (1, 2) 30° about the point (-3, -4) by combining three motions. First, translate (1, 2) by the vector (3, 4). This in effect moves the center of rotation to the origin. Second, rotate the resulting point 30° in the positive (counterclockwise) sense, about the origin. Third, translate the result back by (-3, -4). The 30° rotation about (-3, -4) is the composition of these three isometries represented by the product of three matrices

$$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} & 0 \\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} =$$

$$\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & \frac{3\sqrt{3}-10}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & \frac{4\sqrt{3}-1}{2} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \approx \begin{bmatrix} -2.54 \\ 3.20 \\ 1 \end{bmatrix}.$$

So the coordinates of the result are approximately (-2.54, 3.20).

Similarly, the reflection of a point (2, 1) through the line 2x – y + 3 = 0 can be found by combining three motions. First, translate by the vector (0, -3). This moves the y-intercept to the origin. Second reflect the resulting point through the line 2x – y = 0. Third, translate the result back by the vector (0, 3). The reflection of (2, 1) in the line 2x – y + 3 = 0 is the composition of three isometries and can be represented by the matrix product

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 3\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}-\frac{3}{5} & \frac{4}{5} & 0\\\frac{4}{5} & \frac{3}{5} & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 1 & -3\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}2\\1\\1\end{bmatrix}=$$

$$\begin{bmatrix}-\frac{3}{5} & \frac{4}{5} & -\frac{12}{5}\\\frac{4}{5} & \frac{3}{5} & \frac{6}{5}\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}2\\1\\1\end{bmatrix}=\begin{bmatrix}\frac{-14}{5}\\\frac{17}{5}\\1\end{bmatrix}.$$

Thus the reflection of (2, 1) in the line 2x – y + 3 = 0 is the point (-14/5, 17/5).

The ability to derive the translation, rotation, reflection matrices depends on the mathematical sophistication of the student. For each, the student should know or be in the process of learning the techniques of matrix multiplication and matrix representations of a system of linear equations. The high school student with a thorough knowledge of trigonometric angle sum formulas and some knowledge of vectors will find the derivation of the rotation matrix accessible. The reflection matrix requires an understanding of (1) slope; (2) vector addition; (3) simultaneous solutions to linear equations. Alternative methods of finding the reflection matrix are available and are included in the appendix (Appendices A and B). The translation matrix is probably best left demonstrated for the high school student. It is easy to demonstrate that it works. Why it works is a different matter. The main idea behind homogeneous coordinates is that we are operating in the three dimensional plane z = 1, and all shears of this form in this plane correspond to a two dimensional translation in the same plane.

II. Pedagogical Discussion

A. Secondary School Students

In its simplest form, this topic provides a window into the world of computer graphics for the high school student. By presenting the basic isometries, a student can "move" objects around a grid and gain some feel for the mathematics that makes the magic of computer graphics possible. Thus the exercise provides practice in matrix multiplication in an authentic context.

But the topic can provide much more than an authentic application of matrix multiplication. Even on the procedural level, much more can be gained. For each constructed isometry, there is a homogeneous matrix representation, and vice versa. So in the classroom setting, students can explore the connection using any transformational geometry construction method: a reflecting tool, paper folding, or dynamical geometry software. The student could plot the coordinates of the vertices of a polygon, construct a transformation, and predict the approximate coordinates of the image. The exact coordinates could also be computed using homogeneous coordinates and matrices. The student gains not only practice in matrix multiplication, but also skill with the construction instrument.

On the conceptual level, the student can experience transformations as functions. Discussions of domain, range, and comparisons with single valued functions are appropriate in this setting. One discussion point, for example, is the idea that function composition can be represented by matrix multiplication in this setting. Students can also experience a sense of connection between the geometric and algebraic representation of a transformation, just as they experience the connection between a function of one variable and its graph.

In addition, if the student is allowed to investigate how the isometry matrices are formed, a deep and rich set of connections can be discovered. Considerations of angle addition formulas, positive and negative angles of rotation, slope, standard form of a line, slope-intercept form, slope of a perpendicular line, the distance formula can be employed immediately as one begins to investigate the origins of the matrices involved.

B. Pre-service and in-service teachers.

After the topics in part A have been addressed, university students in a linear algebra course can place these transformations in the context of the full range of transformations in space. Homogeneous coordinates do in fact represent points in the plane z = 1, and homogeneous matrices are special cases of transformation matrices in three dimensions.

For three dimensional objects, homogeneous coordinates are analogous to the two dimensional case. Three dimensional points (x, y, z) are extended to (x, y, z, 1) and the homogeneous matrix for an isometry in three dimensions is extended to a 4x4 matrix. In computer graphics, the three dimensional movement is followed by a projection matrix to the flat screen.

III. Lesson reports and student work

A. Fall semester

The topic was presented as a project topic in an *Introduction to Mathematica* course in the fall of 2001. The expected length of the project was six weeks. In that time, students were to read and understand the mathematical concepts involved (see Appendix A), and then use their knowledge of matrix manipulation and animation in *Mathematica* to produce a program which would perform the reflection, translation, and rotation isometries. Two teams accepted the project. The first team consisted of two in-service secondary math teachers, and the second team consisted of two undergraduate math education majors. Both projects have been included as part of the outcome of the teaching experiment.

The first team of in-service teachers grasped the mathematical concepts of the project early on, and proceeded to develop a program which would demonstrate the three types of isometries. Their main concerns were with programming in *Mathematic*a. They ended the course with a clear presentation that reflected a high degree of expertise in programming *Mathematica*, along with a clear understanding of isometries. One of the teachers reported that she made the same presentation in her high school algebra class.

The team of undergraduates, however, did not completely understand the concepts of the paper, and they did not seek help in understanding. Part of their presentation consisted of duplications of the paper in the text portion of the course, and applying methods not clearly understood.

B. Spring semester

The homogeneous coordinates were presented in a linear algebra course and a college geometry course. In the linear algebra course, the topic was presented in one 50 minute lesson, in the context of studying linear transformations of the plane and space. The majority of the class of 40 students consisted of computer science majors, and a few of them were enrolled in a computer graphics course. The lesson followed the outline of the presentation in Appendix B. First the pre-image was plotted on graph paper, then traced on patty paper. Then the patty paper was moved to the new location by translation, rotation, or reflection and the new coordinates were approximated. The exact location of the image was found using homogeneous matrices and coordinates. The memorable remarks from the computer science students were that they now understood their computer graphics material. Some not taking computer graphics reported that this less theoretical, more concrete approach to linear algebra made the entire course more understandable.

The same format was presented in 75 minutes in the college geometry course. The class consisted of pre-service undergraduate math education majors, in-service teachers, and alternative certification candidates, some of which were already student teaching. The geometry text contained a section on homogeneous coordinates and isometries, into which this lesson was naturally placed. The extra added value of the lesson was the ease in which the concept could be presented to high school students using patty paper, (or dynamical geometry software, or a reflecting tool) to estimate the coordinates and then use homogeneous coordinates to compute the coordinates exactly. The discussion, which followed, focused on the points in the secondary curriculum into which this material could be inserted. The in-service teachers, and one of the undergraduate students from the *Mathematica* project were also in this course. The in-service teachers discussed that they had demonstrated the concept to their own students.

C. Interview with in-service teachers

After the experience in the *Mathematica* class, and the experience in the college geometry class, the in-service teachers were asked to discuss their impressions and implications for high school teaching. They were asked to comment on three aspects of the project: (1) the process of completing the project; (2) how they will demonstrate the lesson in their own classroom; (3) what lesson could they design for their own classes from the what they had learned. They devised a work plan around their busy schedule, and used scheduled class time efficiently to meet and work on the project. The project proceeded from generating the correct numbers, to graphing, animation, and colors. They worked from specific examples to "generic" examples. They broke the project into small pieces: how to graph a line, scaling the axes, reflecting and rotating. Visualization in *Mathematica* helped them to check their results. Obstacles consisted mostly of problems with programming, which was the object of the course. The project goals were clear, but grew over time as they accomplished each task. The professor continued to stretch the students programming ability as they worked on the project.

In their own classrooms, they demonstrated the program to their own students. They talked about asking students to contribute vertices. They showed their program to their students from time to time as they worked on their projects. They talked about visualization for lower level classes. For upper level, they would tried to use it as a motivator, and give the students some of the matrix multiplications to work out.

In an advanced math class, they felt that students would not be able to do enough programming in *Mathematica* to make the project practical in that form. They felt that they could use the LINK and DOWNLOAD feature to put a program into a calculator and let students use it to find images under transformations. They felt that it would not be practical for students to actually write a calculator program themselves. Students' difficulties with programming would interfere with the main lesson, thus confounding the outcome. They also cited logistical problems such as lack of equipment in the classroom, and the *Mathematica* learning curve. They did feel, however, that *Mathematica* would be in the classroom of the future.

D. Understandings gained from the teaching experiments

The undergraduates who were trying to learn matrix algebra in the context of a *Mathematica* programming course had a much more difficult time than the in-service teachers who had some background in linear algebra and experience with matrices in the high school curriculum. For undergraduates, the programming seemed to take precedence over the concept. The same seems to hold true in the secondary school context as well. The comments from the in-service teachers were tempered with considerations of their (secondary) students' programming abilities. If the students have too much trouble programming, then the knowledge gained from the activity will be lost. For this reason, it is probably wiser to contextualize the concept of isometry in the more natural surroundings of a linear algebra or geometry course, for which programming is not the main focus of the activity. In this way, students can focus primarily on the concepts behind isometries in the plane. Students could use the programming abilities which they already have, but keep the focus on the transformational geometry concepts.

Appendix

A. Study document for Fall semester

MATRIX REPRESENTATIONS OF RIGID MOTION IN THE PLANE

We know from the geometry of transformations that a rigid motion, or isometry, in the coordinate plane is a result of a composition of rotation, reflection or translation. The NCTM *Principles and Standards* (2000, p. 314) envisions a clear and broad understanding of the connection between geometric transformations and their matrix representations.

> In high school they will learn to represent these transformations with matrices, exploring the properties of the transformations using both graph paper and dynamic geometry tools. For example, students who are familiar with matrix multiplication could be introduced to matrix representations of transformations….

Discussions of transformations in algebra texts frequently limit the discussion to 2x2 matrix representations of rigid motion. These represent only a few reflections through lines containing the origin, and only rotations about the origin. Yet when we use dynamical geometry applications we see reflections in arbitrary lines, rotations about arbitrary points, and translations by arbitrary vectors. Such actions must have an algebraic underpinning before they can become computer output.

The following will begin with a discussion of the use of 2 x 2 matrices to represent limited types of reflections and rotations in the plane. Secondly, homogeneous coordinates will be introduced to represent translations, as well as reflections and rotations, so that finally, we can develop a method of fully representing the three isometries of reflection, translation, and rotation in a unified way using homogeneous coordinates and matrices.

Reflections And Rotations Using 2x2 Matrices. As shown in the *Principles and Standards* (p. 314), 2x2 matrices can be used to "move" points about in the plane by rotation and reflection, preserving properties of distance and angle measure, therefore congruence. Figure 1, for example, shows the result of reflection of Δ ABD in the line y = 0. The corresponding matrix representation of the reflection is $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, and the matrix representation of the vertices of ΔABD is $\begin{bmatrix} 1 & 2 & 4 \\ 0 & 5 & 2 \end{bmatrix}$, where the coordinates of A(1, 0) are written in column form as $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, etc. Thus the outcome of the matrix product

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 4 \\ 0 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & -5 & -2 \end{bmatrix}$$

reveals the coordinates of $\Delta A'B'D'$ which are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 2 \\ -5 \end{bmatrix}$, and $\begin{bmatrix} 4 \\ -2 \end{bmatrix}$, respectively.

Similarly, rotations centered at the origin can be performed using the rotation matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. For example the parallelogram ABCD can be rotated 90° about the origin to $A''B''C''D''$ with coordinates found by

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 5 & 4 \\ 0 & 5 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -5 & -7 & -2 \\ 1 & 2 & 5 & 4 \end{bmatrix}.$$

The approximate coordinates of A'', B'', C'' and D'', respectively, are $A''\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $B''\begin{bmatrix} -5 \\ 2 \end{bmatrix}$, $C''\begin{bmatrix} -7 \\ 5 \end{bmatrix}$, and $D''\begin{bmatrix} -2 \\ 4 \end{bmatrix}$ as shown in Figure 2.

Translations. While rotation and reflection can be represented by 2x2 matrix multiplication, translation cannot. Suppose T is a translation by vector <h, k>. Then T would take (x, y) into (x+h, y+k). For example, translating A(1, 0) by vector <2, 3> would result in $A'(3,3)$. The matrix representation of this translation is therefore

$$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$
$$= \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

which is matrix addition, not multiplication.

There is a simple way, however, of unifying the three basic isometries of rotation, reflection, and translation under matrix multiplication. It is to introduce the idea of homogeneous coordinates of points in the plane, and homogeneous matrices representing the basic isometries.

Homogeneous Coordinates. A homogeneous coordinate is simply a 3 x 1 column matrix formed by placing a 1 in the third row. For example the point B (2, 5) which was represented by the column matrix $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$ is now represented by the column matrix $\begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$. The extension from 2x1 to a 3x1 column matrix would necessitate the extension of the transformation matrix to 3x3 as well. Therefore, the homogeneous matrix representation of a transformation must be a specially designed 3 x 3 matrix. The following example will illustrate how the extension to a 3 x 3 matrix takes place for simple rotations and reflections. The 2 x 2 matrix representing a rotation in for 2x2 matrices is expanded from $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ to the 3 x 3 matrix $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$. It is also easy to translate the reflections from 2 x 2 matrices to 3 x 3 matrices. For example, the previous reflection of the triangle ΔABC would now be represented by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 4 \\ 0 & 5 & 2 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & -5 & -2 \\ 1 & 1 & 1 \end{bmatrix}.$$

Notice that the third row does not correspond physically to a coordinate.

The real advantage of homogeneous coordinates and homogeneous matrices is in the representation of translation. Whereas before, for example, the translation which carried (1, 0) to (3, 5) was represented by the matrix addition $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$, it can now be represented by multiplication using homogeneous representation.

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$$

What happened? The third column of the matrix incorporates the translation vector. Now all three basic isometries can be represented by matrix multiplication. Because we can now use the same matrix multiplication to translate points, we have the full range of possible isometries available to us symbolically. We are no longer restricted to rotating figures about the origin. The following example will demonstrate how we can use matrices to rotate a figure about an arbitrary point.

Rotation about an Arbitrary Center. Suppose we want to rotate point B $\begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$ through an angle of 90° about the point C $\begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$, as shown in figure 3 . First, translate point B by $\begin{bmatrix} -4 \\ -2 \\ 1 \end{bmatrix}$ (figure 4a).

$$\begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$$

This has the effect of moving the center C of rotation to the origin. Second, rotate the result by 90° (figure 4b).

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix} = \left[\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right] \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}$$

Lastly, translate the figure back by $\begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$ (figure 4c):

$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Of course, the product of the three matrices on the left side is $\begin{bmatrix} 0 & -1 & 6 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix}$,

which is the particular transformation matrix which will rotate any point 90° about the point (4,2). In particular, ΔDEF will rotate to $\Delta D'E'F'$ as shown in Figure 5

Reflection Through An Arbitrary Line. Just as we are no longer restricted to the origin as the center of rotation, we can also expand our capabilities for reflections. Suppose we have the point $P(4,1)$ and the line L: $x-3y+9=0$, as shown in Figure 6. We wish to find the reflection of *P* in the line: that is, we wish to find point P' on the line perpendicular to $x-3y+9=0$, which is the same distance from line L as *P*. The following strategy will produce P: (1) find the line through P which is perpendicular to line L; (2) find the point of intersection, Q, of L with the perpendicular line; (3) Find the vector $\overrightarrow{PQ}$; and (4) find P' by vector addition $\overrightarrow{OP}+2\overrightarrow{PQ}$.

The line perpendicular to line L through $P(4,1)$ is $y-1=-3(x-4)$ which simplifies to the equivalent form $3x+y=13$. We can now find the point of intersection of the line and its perpendicular by solving the system:

$$x-3y=-9$$
$$3x+y=13$$

to produce the intersection point $Q(3,4)$.

In the language of vectors, we now wish to add $\overrightarrow{OP}+2\overrightarrow{PQ}$ to produce the vector $\overrightarrow{OP'}$. Using $\overrightarrow{OP}=\langle 4,1\rangle$ and $\overrightarrow{PQ}=\langle -1,3\rangle$, we find $\overrightarrow{OP'}=\langle 2,7\rangle$, which corresponds to the correct reflection of point P in the line L.

We can do the same as above in the general case, beginning with the line $Ax+By+C=0$ and the point $P(x_0,y_0)$. The line through P perpendicular to $Ax+By+C=0$ is the line $y-y_0={}^{B}\!/\!_{A}(x-x_0)$, or equivalently $-Bx+Ay=Ay_0-Bx_0$. Solving

$$\begin{aligned} Ax+By&=-C \\ -Bx+Ay&=Ay_0-Bx_0 \end{aligned}$$

yields the point of intersection $Q(\frac{B^2x_0-ABy_0-AC}{A^2+B^2},\frac{-ABx_0+A^2y_0-BC}{A^2+B^2})$. Vectors $\overrightarrow{PQ}$ and $\overrightarrow{OP}$ take on the following forms: $\overrightarrow{PQ}=\left\langle \dfrac{-A^2x_0-ABy_0-AC}{A^2+B^2},\quad \dfrac{-ABx_0-B^2y_0-BC}{A^2+B^2}\right\rangle$, $\overrightarrow{OP}=\langle x_0,y_0\rangle$. So the vector representation of the reflection is $\overrightarrow{OP'}=\overrightarrow{OP}+2\overrightarrow{PQ}$. Therefore,

$$\overrightarrow{OP'}=\left\langle \frac{(B^2-A^2)x_0-2ABy_0-2AC}{A^2+B^2},\quad \frac{-2ABx_0+(A^2-B^2)y_0-2BC}{A^2+B^2}\right\rangle.$$

We can write the x and y coordinates of P' in equation form:

$$\begin{aligned} x&=\tfrac{B^2-A^2}{A^2+B^2}x_0-\tfrac{2AB}{A^2+B^2}y_0-\tfrac{2AC}{A^2+B^2} \\ y&=\tfrac{-2AB}{A^2+B^2}x_0+\tfrac{A^2-B^2}{A^2+B^2}y_0-\tfrac{2BC}{A^2+B^2} \end{aligned}$$

from which we can produce the matrix multiplication representation of P'.

$$P'=\begin{bmatrix} \frac{B^2-A^2}{A^2+B^2} & \frac{-2AB}{A^2+B^2} & \frac{-2AC}{A^2+B^2} \\ \frac{-2AB}{A^2+B^2} & \frac{A^2-B^2}{A^2+B^2} & \frac{-2BC}{A^2+B^2} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_0 \\ y_0 \\ 1 \end{bmatrix}.$$

Using this form, the reflection matrix for the line L: $x-3y+9=0$ (Figure 6) is $\begin{bmatrix} \frac{4}{5} & \frac{3}{5} & \frac{-9}{5} \\ \frac{3}{5} & \frac{-4}{5} & \frac{27}{5} \\ 0 & 0 & 1 \end{bmatrix}$. The reflection of (4, 1) in line L is computed by matrix multiplication:

$$\begin{bmatrix} \frac{4}{5} & \frac{3}{5} & \frac{-9}{5} \\ \frac{3}{5} & \frac{-4}{5} & \frac{27}{5} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ 1 \end{bmatrix}.$$

Using these forms, we can now (1) rotate a figure about *any* point in the plane, (2) translate by *any* vector, and we can (3) reflect through *any* line in the plane with the ease of matrix multiplication. Therefore any constructed isometry in the plane can be represented algebraically using homogeneous coordinates.

References.

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Faux, I. D., and M. J. Pratt. *Computational Geometry for Design and Manufacture.* Chichester, England: Ellis Horwood Limited, 1983

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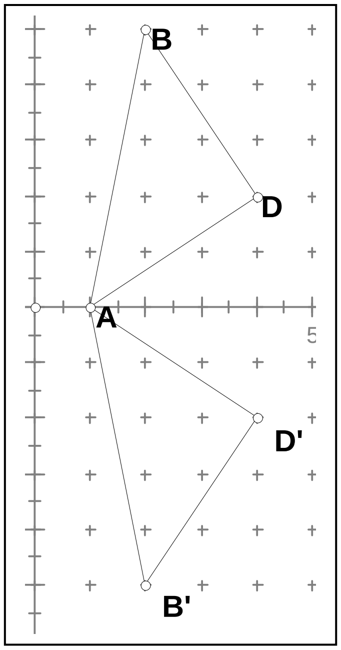


Figure 1: Reflection in the line y = 0

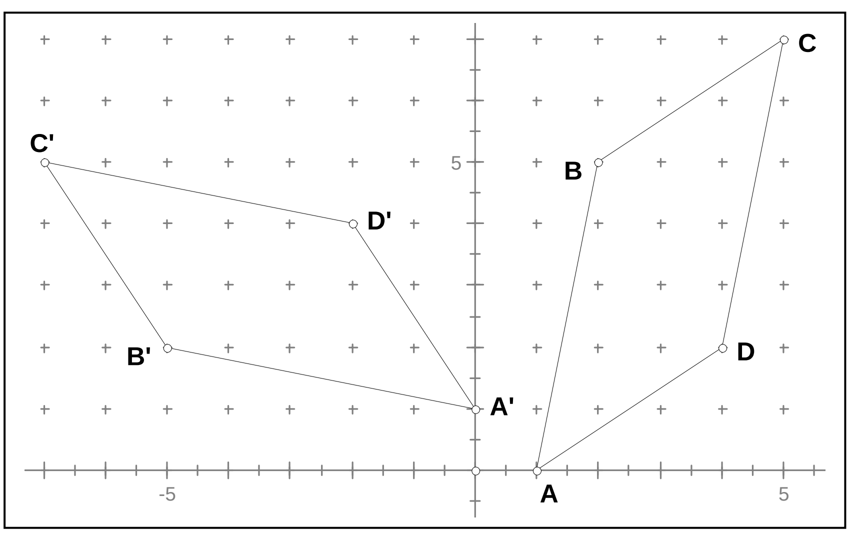


Figure 2: 90° Rotation about the origin.

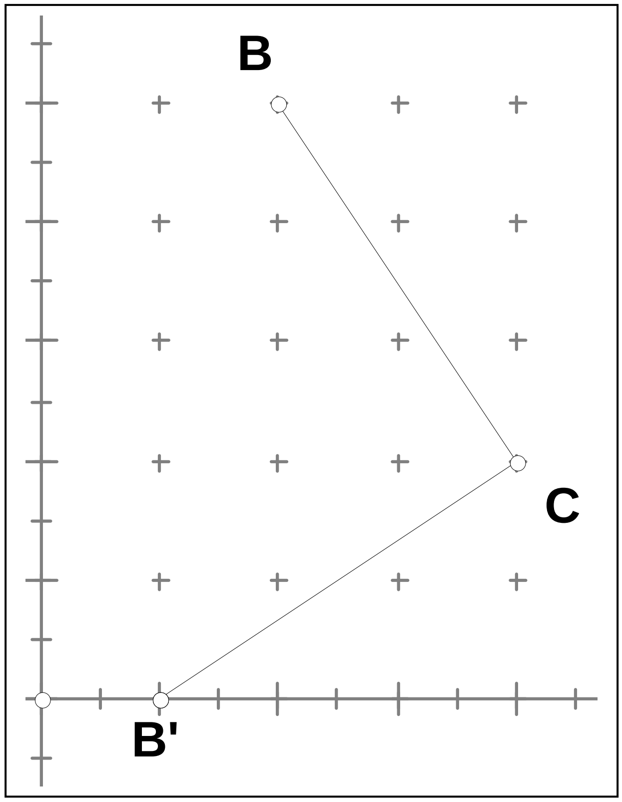


Figure 3: Center of rotation C (4, 2)

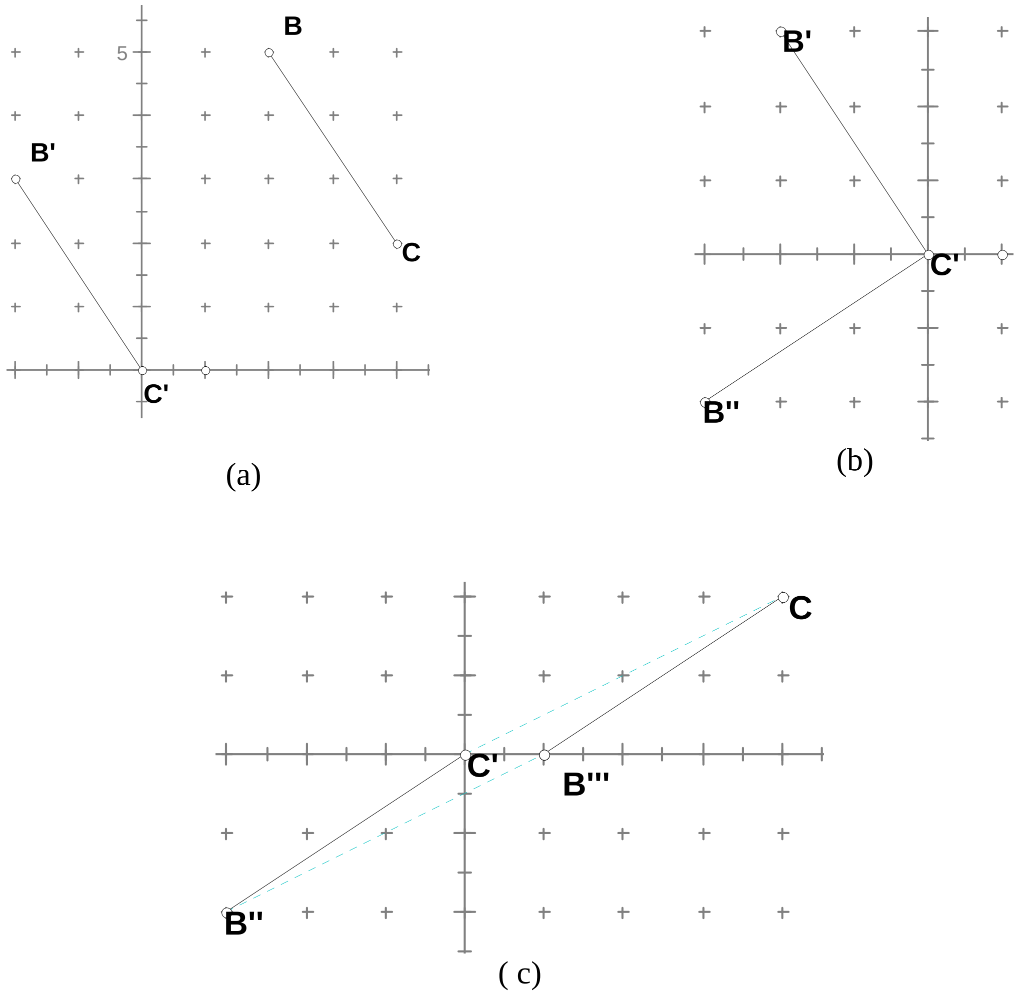
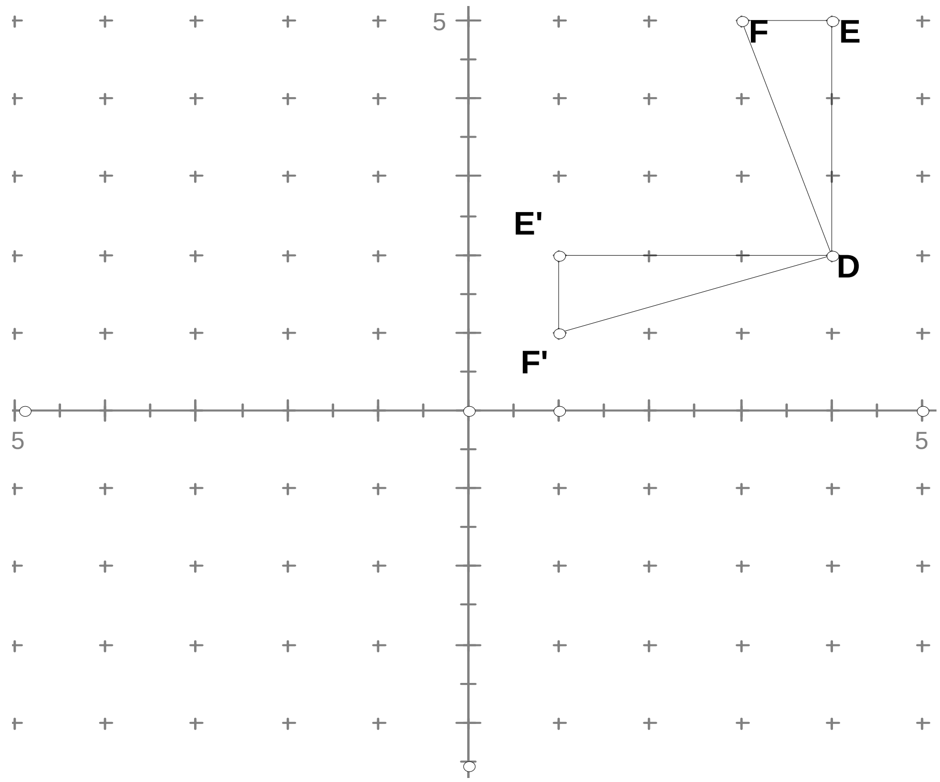
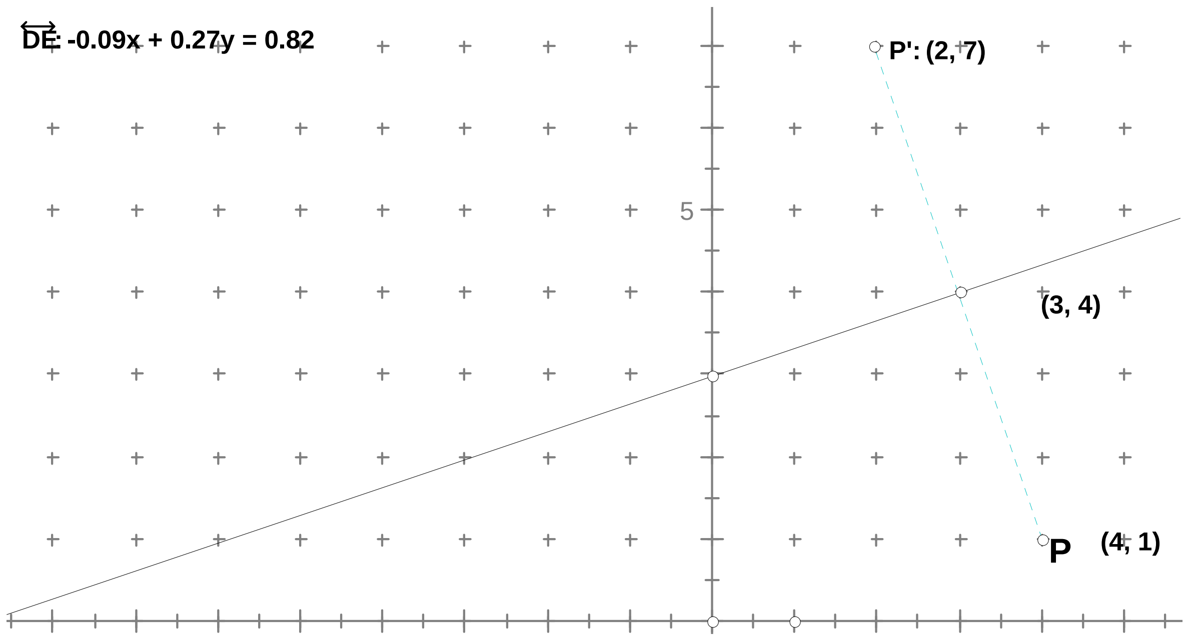


Figure 4: (a) Translation, (b) Rotation, (c) Translation.



$$\begin{bmatrix} 0 & -1 & 6 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 4 & 3 & 4 \\ 2 & 5 & 5 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

Figure 5. Rotation of Δ DEF about D(4, 2)

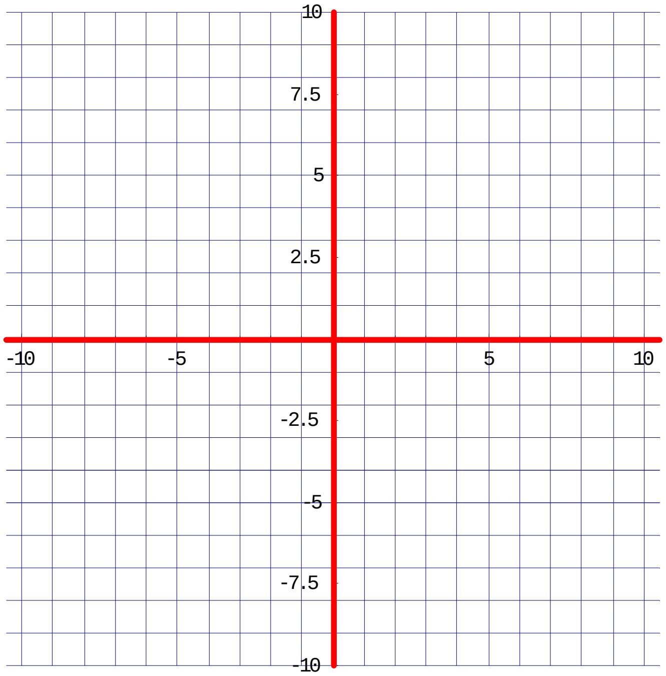


$$\begin{bmatrix} \frac{4}{5} & \frac{3}{5} & \frac{-9}{5} \\ \frac{3}{5} & \frac{-4}{5} & \frac{27}{5} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ 1 \end{bmatrix}$$

Figure 6. Reflection of P in line L: x – 3y + 9 = 0

B. Lesson Activities for Spring semester.

REFLECTION IN THE X-AXIS



Plot the points A(2,1), B(5,1), and C(4,3) on the grid shown.

Then reflect $\triangle ABC$ in the x-axis. What are the reflected points? [A'(2, -1), B'(5, -1), and C'(4, -3).]

So if P is a generic point in the plane, what are the coordinates of the reflected point P'?

$$x' = x$$

$$y' = -y$$

or

$$x' = 1 \cdot x + \ 0 \cdot y$$

$$y' = 0 \cdot x + (-1) \cdot y$$

so

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Notice that each point (x, y) is represented by the column matrix $\begin{bmatrix} x \\ y \end{bmatrix}$

Notice also that the three reflections

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix},$$

and

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

could be consolidated into one matrix multiplication

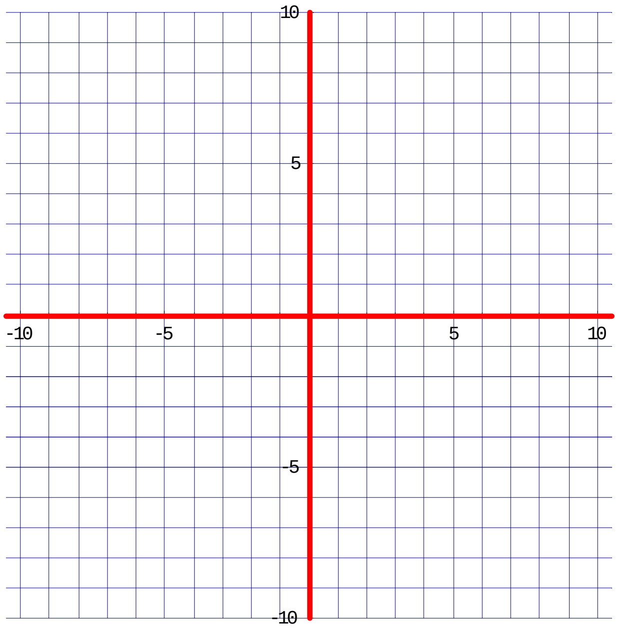
$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 5 & 4 \\ 1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 5 & 4 \\ -1 & -1 & -3 \end{bmatrix}.$$

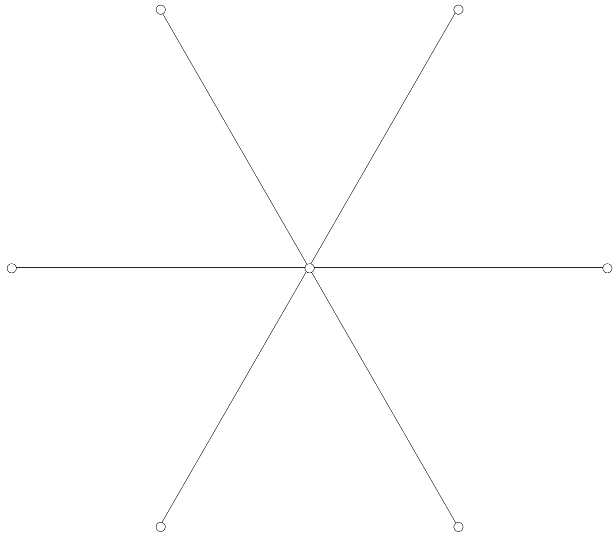
So we can describe the reflection of $\triangle ABC$ in the following way:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \left[A \,|\, B \,|\, C \right] = \left[A' \,|\, B' \,|\, C' \right],$$

where A, B and C are columns which represent the vertices of Δ ABC.

ROTATIONS





Use patty paper to copy the "protractor" above. Trace Δ ABC below and then rotate the triangle 60° and estimate the coordinates of the vertices.

The matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ will rotate a figure through a positive angle θ centered at the origin. For example, a 60° rotation about (0,0) will result in a matrix $\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$ and a rotation of Δ ABC will look like:

$$\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 2 & 5 & 4 \\ 1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{2-\sqrt{3}}{2} & \frac{5-\sqrt{3}}{2} & \frac{4-3\sqrt{3}}{2} \\ \frac{2\sqrt{3}-1}{2} & \frac{5\sqrt{3}-1}{2} & \frac{4\sqrt{3}+3}{2} \end{bmatrix}$$

, which, with a hand held calculator, you can approximate and plot as $\begin{bmatrix} 0.1 & 1.6 & -0.6 \\ 2.2 & 4.8 & 5 \end{bmatrix}$.

TRANSLATIONS

Notice that the rotations and reflections we have encountered so far leave the origin (0, 0) fixed. Any 2x2 matrix will leave the origin fixed. So it's impossible to represent a translation with 2x2 matrix multiplication. We can get around this problem, however, if we take a "slice" of three dimensional space, and represent our points on the plane z = 1. So now we will continue with representing translations, rotations, and reflections using 3x3 matrices and 3x1 representations of points, in the plane z = 1.

HOMOGENEOUS COORDINATES AND MATRICES

To achieve a complete representation of isometries in the plane, we expand the representations of points, the rotation and reflection matrices we have found so far, and introduce a very nice representation of translations in the plane.

Points in the Plane

Represent each point $\begin{bmatrix} x \\ y \end{bmatrix}$ as the 3x1 column matrix $\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$. So, for example,

A(2,1) is represented by $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, B(5,1) is represented by $\begin{bmatrix} \\ \\ 1 \end{bmatrix}$, and C(4,3) is represented by $\begin{bmatrix} \\ \\ 1 \end{bmatrix}$.

Reflections

Reflection in the x-axis: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ is expanded to the 3x3 matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Reflection in the y-axis: $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ is expanded to the 3x3 matrix $\begin{bmatrix} & & 0 \\ & & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Reflection in the line y = x: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is expanded to $\begin{bmatrix} & & 0 \\ & & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Rotations:

A rotation $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is similarly expanded to $\begin{bmatrix} & & 0 \\ & & 0 \\ 0 & 0 & 1 \end{bmatrix}$

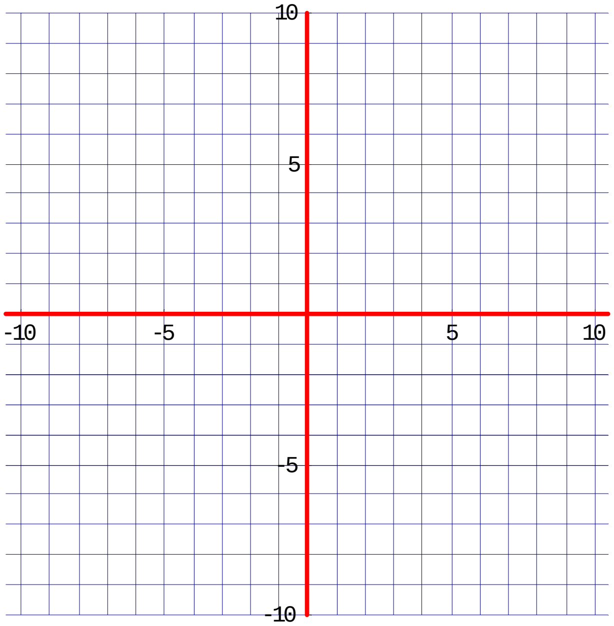
Translations

Translations have no 2x2 matrix multiplication representation, because every translation matrix would necessarily move each point in the plane the same distance, and we know that matrix multiplication leaves the origin (0, 0) fixed. But we can represent translations with 3x3 matrices.

A translation which moves each point a distance (h, k), for example, can be represented by the matrix

$$\begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix}.$$

TRANSLATIONS



For example, to translate Δ ABC by (-5, 4) graphically.
First, use patty paper to trace Δ ABC: A(2,1), B(5,1), and C(4, 3), AND the origin (0,0).

Then, without rotation, move each point by (-5, 4), so that the origin (0, 0) is mapped to the point (-5, 4).

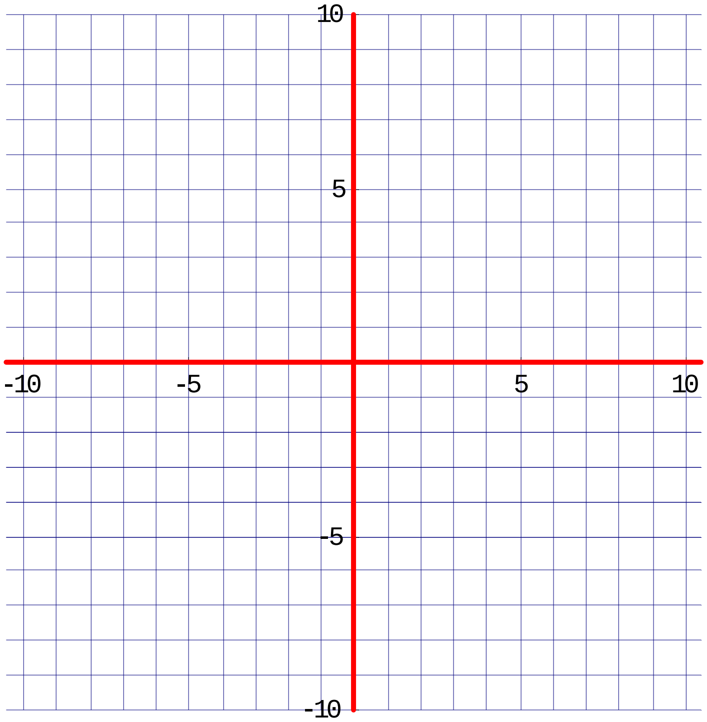
and the new triangle Δ A'B'C' will have coordinates A'(-3,5), B'(0, 5), and C'(-1, 7) respectively.

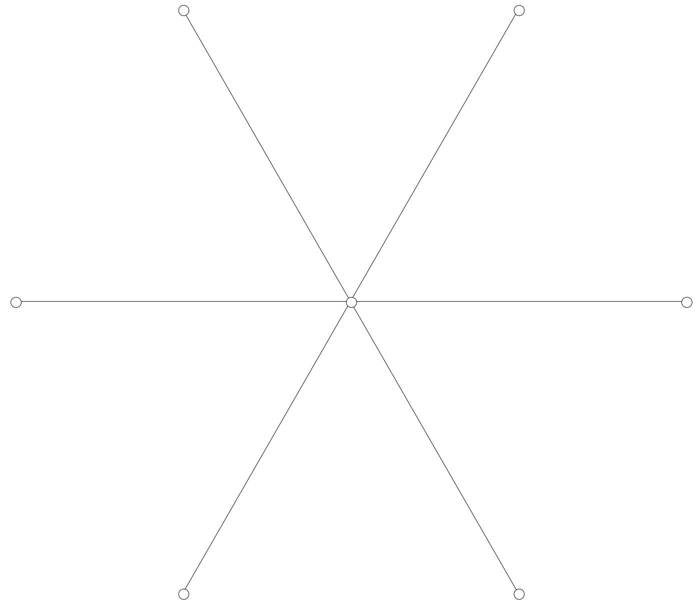
If we multiply the old coordinates by the translation matrix:

$$\begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 5 & 4 \\ 1 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ 1 & 1 & 1 \end{bmatrix}$$

we can compute the new vertices.

ROTATION ABOUT AN ARBITRARY POINT





Use another copy of your patty paper "protractor." Place the center of rotation at the point (-2, -5) and copy the triangle. Rotate 60° about the point D(-2, -5). Estimate the coordinates of the rotated triangle Δ A'B'C'. Copy your rotated triangle on the grid.

The matrix of rotation is a product of three simple movements. First, we move the center of rotation, D to the origin, (0,0). Second, we then rotate the figure 60° about the origin. Third, we move the center of rotation back to point D(-2, -5).

First: Translate your figure by (2, 5). This will move point D to the origin. Note the coordinates of the triangle vertices: A' (,), B' (,), and C' (,). Now use the translation matrix to compute the same coordinates

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 5 & 4 \\ 1 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad \\ & & \\ 1 & 1 & 1 \end{bmatrix}$$ translates Δ ABC and the center of rotation D so that the center of rotation is the origin, (0,0).

Next, rotate the figure 60° about the origin. Estimate the coordinates of the vertices again, and compute :

$$\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 4 & 7 & 6 \\ 6 & 6 & 8 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ 1 & 1 & 1 \end{bmatrix}$$

$$\approx \begin{bmatrix} -3.2 & -1.7 & -3.9 \\ 6.5 & 9.1 & 9.2 \\ 1 & 1 & 1 \end{bmatrix}$$

rotates the new triangle 60° about (0,0).

Lastly, we will translate the center back to D(-2, -5), using the translation matrix

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2-3\sqrt{3} & \frac{7}{2}-3\sqrt{3} & 3-4\sqrt{3} \\ 2\sqrt{3}+3 & \frac{7\sqrt{3}}{2}+3 & 3\sqrt{3}+4 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -3\sqrt{3} & \frac{3}{2}-3\sqrt{3} & 1-4\sqrt{3} \\ 2\sqrt{3}-2 & \frac{7\sqrt{3}}{2}-2 & 3\sqrt{3}-1 \\ 1 & 1 & 1 \end{bmatrix},$$

$$\approx \begin{bmatrix} -5.2 & -3.7 & -5.9 \\ 1.5 & 4.1 & 4.2 \\ 1 & 1 & 1 \end{bmatrix}$$

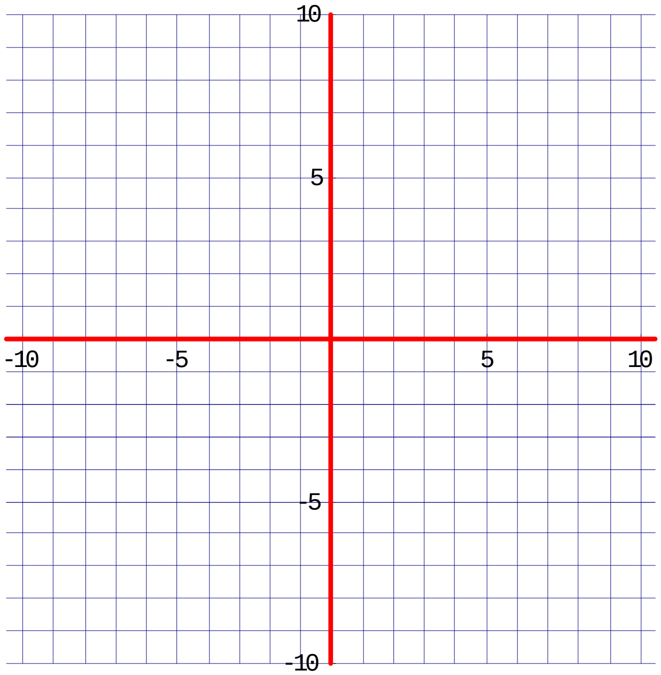
which produces the desired rotation.

In general, then, a rotation about any point D(h, k) through an angle θ can be accomplished by the product of three matrices:

$$\begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -h \\ 0 & 1 & -k \\ 0 & 0 & 1 \end{bmatrix}$$

In our case, $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} & \frac{-5\sqrt{3}}{2}-1 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & \sqrt{3}-\frac{5}{2} \\ 0 & 0 & 1 \end{bmatrix}$

REFLECTION IN AN ARBITRARY LINE: Ax + By + C = 0, A>0



The line in this example has the equation $2x - y + 3 = 0$, and our same triangle, Δ ABC, is shown.

Trace the line, the x-intercept, the y-intercept, and the triangle on a sheet of patty paper. Carefully turn the patty paper over and let the line fall back on itself, matching the x- and y-intercepts. Trace the reflected triangle, and estimate the coordinates of the vertices.

A' (,), B' (,), C'(,)

There are several ways of finding the reflection in an arbitrary line. We will use two observations about lines:

First observation: The slope of the line is –A/B. The angle of intersection, θ, of the line with the x-axis has the property tan θ = -A/B. So $\sin\theta = \frac{A}{\sqrt{A^2+B^2}}$, $\cos\theta = \frac{-B}{\sqrt{A^2+B^2}}$.

Second observation: A non-vertical line has a y-intercept, -C/B.

So we will use a series of moves to reflect the Δ ABC in the line $2x - y + 3 = 0$.

First, translate the y-intercept, (0, 3) to the origin (0,0) : $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 5 & 4 \\ 1 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 5 & 4 \\ -2 & -2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

Next, use patty paper to rotate by -θ so that the reflecting line is horizontal, and on the x-axis. Estimate the coordinates of the rotated triangle. A'(,), B'(,) , C' (,)

Note that $\cos\theta = \frac{1}{\sqrt{5}}$, and $\sin\theta = \frac{2}{\sqrt{5}}$, so the rotation matrix is: $\begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

And $\begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 5 & 4 \\ -2 & -2 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{4}{\sqrt{5}} \\ \frac{-6}{\sqrt{5}} & \frac{-12}{\sqrt{5}} & \frac{-8}{\sqrt{5}} \\ 1 & 1 & 1 \end{bmatrix}$

$$\approx \begin{bmatrix} -0.9 & 0.4 & 2 \\ -3 & -5 & -4 \\ 1 & 1 & 1 \end{bmatrix}$$

Now

the reflecting line is the x-axis. So use the reflection $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ to reflect the triangle over the line:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{4}{\sqrt{5}} \\ \frac{-6}{\sqrt{5}} & \frac{-12}{\sqrt{5}} & \frac{-8}{\sqrt{5}} \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{4}{\sqrt{5}} \\ \frac{6}{\sqrt{5}} & \frac{12}{\sqrt{5}} & \frac{8}{\sqrt{5}} \\ 1 & 1 & 1 \end{bmatrix}$$

$$\approx \begin{bmatrix} -0.9 & 0.4 & 2 \\ 3 & 5 & 4 \\ 1 & 1 & 1 \end{bmatrix}$$

Now reverse the rotation and the translation:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} & 0 \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{4}{\sqrt{5}} \\ \frac{6}{\sqrt{5}} & \frac{12}{\sqrt{5}} & \frac{8}{\sqrt{5}} \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{-14}{5} & \frac{-23}{5} & \frac{-12}{5} \\ \frac{17}{5} & \frac{29}{5} & \frac{31}{5} \\ 1 & 1 & 1 \end{bmatrix}$$

Essentially, the transformation matrix that produces the reflection across the line Ax+By+C = 0, A>0, is:

[Translation by (0, C/B)]$^{-1}$[Rotation by -θ]$^{-1}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ [Rotation by -θ] [Translation by (0, C/B)].

For our line $2x - y + 3 = 0$, that matrix would be: $\begin{bmatrix} \frac{-3}{5} & \frac{4}{5} & \frac{-12}{5} \\ \frac{4}{5} & \frac{3}{5} & \frac{6}{5} \\ 0 & 0 & 1 \end{bmatrix}$