Descartes and Differential Equations
A high-school math problem that invites in-depth analysis

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Introduction

Dick Stanley and Pat Callahan have developed an approach to teaching mathematics to prospective and practicing high-school teachers that involves taking a simple problem, examining and comparing multiple solutions and generalizing the problem in numerous ways, [1]. They pay careful attention to the different representations (numerical, algebraic, geometric, *etc.*) that are employed in different contexts and the ways in which they are systematically related to one another. Their approach often demands a lot of patience, yet it seems successful in cultivating habits of mind that mathematicans value but which often seem missing from high school math.

The "evaporation problem" is a typical example:

> *A solution of salt in water is* 99% *water. What proportion of the water must evaporate in order for the percentage of water to decrease to* 98%*?*

The problem can be represented in a number of different ways. The most naive students find it useful to make a table showing various quantities and percentages. Other than the numerical approach, there are some simple *ad hoc* ways of representing the problem that give the solution very quickly. *Example*: "We must double the concentration of salt, so we need to cut the total amount of solution in half. Thus, 50/99 of the water originally present must go."

Algebraic solutions in which the quantities involved become variables and in which the problem is rephrased by means of equations are the natural and expected approach for mathematically mature students. The algebraic approach leads to generalizations in which *the amount of salt, the amount of water removed and the initial and final concentrations of salt all become variables.* Any given concrete instance of the problem, then, corresponds to a consistent choice of values for the four variables. Such a choice is a point on the two-dimensional manifold $M \subseteq \mathbf{R}^4$ given by the equations:

$$x = \frac{r}{1-s} \quad \text{and} \quad y = \frac{s}{1-r}.$$

Here, we are assuming we have started with 1 unit of solution; s denotes the amount of salt present (in the original problem, .01 units), r denotes the amount of water removed, x denotes the proportion of water removed and y denotes the resulting concentration of salt. Every one of the variables must be between 0 and 1, so the instances of the problem correspond the points on M that lie inside a hypercube. Any two of the variables determine the other two. The equations defining M have an obvious symmetry. This suggests further generalizations involving mixtures with multiple components.

Another good example is the car catch-up problem, which is described in detail in [2]. If I head out of town travelling at 75 miles per hour and the police begin pursuit 10 mintes later at 85 miles per hour, how long does it take them to catch up? In the process of generalizing, all the amounts become variables. A graphical representation of the problem shows that it is essentially the same as the "sea-island problem", in which we determine the height of an off-shore peak by measuring the angle of elevation observed from two points on the shore.

The success of the didactic method is dependent on the quality of the the initial problem. It's true that a good mathematical imagination can trace a path from almost any problem into mathematically interesting territory, but it's equally true that some situations are just naturally very rich in opportunities. The two problems already cited are fine examples.

Below, we present some of geometry problems that were originally suggested to us by a device that Descartes describes in Book II of his *Geometry*; see page 46 of [3]. The attraction of these problems is that they can be represented in a very simple physical device that anyone can make from two sheets of paper. Yet they have profound mathematical connections. Indeed, the third problem has proved beyond our immediate ability to solve.

As the title of this paper suggests, this is not a classroom-ready lesson-plan. Our intention here is to describe some geometry problems that can easily be posed to high-schoolers, which can be solved by high-school methods yet which have profound connections with deeper, more advanced math. We hope that others might use this as a first step toward the development of a set of activities that would be useful in the professional training of high-school teachers or even in developing some high-school curriculum.

A geometry problem

Draw x- and y-axes on a piece of paper and label the origin A. Take another sheet of paper with two edges that form a right angle at corner B. Place the second sheet on the first so that: *i*) B is in the first quadrant, *ii*) the first edge extending from B passes through A and *iii*) the other edge extending from B passes through the positive x-axis. Let C denote the point where the second edge and the x-axis meet.

Question 1. *Retaining the three conditions, how do we move B so that C remains fixed?*

This can be answered by means of a theorem attributed to Thales. Let AC be a diameter of a circle γ and let B be a point distinct from A and C. If B is on γ, then $\angle ABC$ is right. Thales' Theorem has a converse, which is not hard to demonstrate: if $\angle ABC$ is right, then B is on γ. The converse of Thales' Theorem provides an immediate answer to Question 1 is: B moves on the circle with diameter AC.

Question 1 can also be addressed algebraically. Let x and y be the coordinates of B, *i.e.*, $B = P(x, y)$. By any of several methods, it can be deduced that

$$C = ((x^2 + y^2)/x, 0).$$

For example, let $D := (x, 0)$ be the base of the altitude of $\triangle ABC$ from B. Then $|DC|/y = y/x$, so $|DC| = y^2/x$ and hence $|AC| = (x^2 + y^2)/x$. The condition that C remains fixed is:

$$(x^2 + y^2)/x = 2k\ , \quad k \text{ constant.} \tag{1}$$

This is equivalent to:

$$y^2 + (x-k)^2 = k^2,$$

which means that $B = P(x, y)$ lies on the circle with diameter AC. For future reference, note that when we differentiate (1) we get:

$$\frac{dy}{dx} = \frac{-(x^2 - y^2)}{2\,x\,y}. \tag{1'}$$

Question 2. *Retaining the three conditions, how do we move B so that C moves as rapidly as possible? (This can be rephrased more precisely as follows. View the position of B as a function of time $B(t)$. We want to find the particular function B_0 that maximizes the ratio of speeds $|C'(t)|\,/\,|B'(t)|$.)*

We'll appraoch this problem first by means of calculus. After this, we'll describe an elementary geometric solution. Assume that B moves at unit speed, so $B' = (x', y') = (\cos\theta, \sin\theta)$, where θ depends on t. We want to find the path that B must follow to maximize the speed of C:

$$|C'| = \left|\frac{(x^2 - y^2)\,x' + 2\,x\,y\,y'}{x^2}\right| = \left|\frac{(x^2 - y^2)\,\cos\theta + 2\,x\,y\,\sin\theta}{x^2}\right|.$$

The extereme points of the function $f(\theta) := a\cos\theta + b\sin\theta$ occur when $\tan\theta = b/a$, and thus $|C'|$ has its extreme values when $\tan\theta = \frac{2\,x\,y}{x^2 - y^2}$. Since $\tan\theta$ is the slope of the tangent line to the trajectory of B, to maximize the speed of C, B should move on a trajectory so that

$$\frac{dy}{dx} = \frac{2\,x\,y}{x^2 - y^2}. \tag{2}$$

Comparing to $(1')$, we see that this means that B should move in a direction perpendicular to the path that would leave C fixed. To solve this differential equation, set $y = xw$, and this becomes

$$x + x\frac{dw}{dx} = \frac{2w}{1 - w^2},$$

which simplifies to

$$\frac{1}{x} = \frac{1 - w^2}{w(1 + w^2)}\frac{dw}{dx}.$$

Integrating, and assuming x and y positive

$$\ln x + C = \ln\,\frac{w}{1 + w^2},$$

or

$$Kx = \frac{w}{1 + w^2} = \frac{xy}{x^2 + y^2}.$$

Seting $2k = 1/K$, this simplifies to

$$x^2 + (y-k)^2 = k^2.$$

Thus, the trajectory is along a circle with center on the y-axis.

Like Question 1, Question 2 can be addressed by means of an elementary geometric argument. Let E denote the center of the circle through A, B and C. E is the point where the perpendicular bisector of AB meets AC. To maximize the speed of C is to make the radius of this circle increase as rapidly as possible. To do this, B should move parallel to EB. With a little algebra, one can show that the slope of EB is $\frac{2\,x\,y}{x^2-y^2}$, so this provides another way to demonstrate that (2) is a necessary condition. Of course, this only describes the local condition that the maximizing trajectory must follow. To find the global solution, we need a more powerful argument. Let F be the point where the perpendicular bisector of AB meets the y-axis, and let M be the midpoint of AB. Then, $\triangle AME \cong \triangle BME$ and $\triangle AMF \cong \triangle BMF$. It follows that $\angle FBE$ is right. Thus, if B travels along the cirle with fixed center F, its direction will always be orthogonal to the circle through A, and the moving points B and C.

For dessert, here is a question that I am not presently able to answer:

Question 3. *If B moves at constant speed, what trajectory must it follow so that C also moves at constant speed?*

One solution is clearly for B to move along a line through A. Other paths are determined by different speed ratios and initial positions.

References

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[2] Stanley, D. and Walukiewicz, J., In-depth mathematical analysis of ordinary high school problems, Mathematics Teacher vol. 97, no. 4, April 2004, pp. 248–255.

[3] Smith, D. E., and Latham, M. L., translators, The Geometry of Rene Descartes, Dover.