Monoreflections of archimedean ℓ-groups, regular σ-frames and regular Lindelöf frames

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Abstract. We prove that in the category of achimedean lattice-ordered groups with weak unit there is no homomorphism-closed monoreflection strictly between the strongest *essential* monoreflection (the so-called "closure under countable composition") and the strongest monoreflection (the epicompletion). It follows that in the category of regular σ-frames, the only non-trivial monoreflective subcategory that is hereditary with respect to closed quotients consists of the boolean σ-algebras. Also, in the category of regular Lindelöf locales, there is only one non-trivial closed-hereditary epi-coreflection. The proof hinges on an elementary lemma about the kinds of discontinuities that are exhibited by the elements of a composition-closed l-group of real-valued functions on $\mathbb{R}$.

0. Introduction.

W denotes the category of archimedean ℓ-groups (*i.e.*, lattice-ordered groups) with distinguished weak order unit and unit-preserving ℓ-homomorphisms. If $\mathcal{X}$ is any locale, $C(\mathcal{X})$ denotes the ℓ-group of all continuous real-valued functions on $\mathcal{X}$. We view $C(\mathcal{X})$ as an object of **W** using the ℓ-group operations naturally induced by the reals. The constant function 1 plays the role of the weak unit. **CCC** denotes the full subcategory of **W** whose objects of are those ℓ-groups that are isomorphic to some $C(\mathcal{X})$.

What is the relation of **CCC** to the larger category **W**? First, **CCC** is H-closed in **W**, *i.e.*, hereditary with respect to quotients, *i.e.*, any homomorphic image in **W** of any object of **CCC** is itself in **CCC**; see 1.3, below. Second, **CCC** is monoreflective in **W**, *i.e.*, each **W** object A has a functorial hull γA in **CCC**; see §1 for detailed definitions. Third, γA is an essential extension of A, *i.e.*, if $K \subseteq \gamma A$ is the kernel of a **W**-morphism and $K \cap A = \{0\}$, then $K = \{0\}$. Fourth, **CCC** is the *smallest* monoreflective subcategory of **W** *for which the reflection morphisms are essential* [BH2]. More information may be found in [H1], [H2] and [MV2].

An object B of **W** is called epicomplete if every **W**-epimorphism with domain B is surjective. Let **E** denote the full subcategory of **W** comprising the epicomplete objects. Then $\mathbf{E} \subseteq \mathbf{CCC}$. In [MV2] and [BH], we showed that **E** is in fact the *smallest of all* monoreflective subcategories of **W**. Like **CCC**, **E** is H-closed, but unlike **CCC** the reflection morphisms to **E** need not be essential.

In the present paper, we prove the following theorem, which shows how **CCC** and **E** are related:

Main Theorem. *There is only one non-trivial H-closed monoreflective subcategory in* **CCC**, *namely* **E**.

The theorem is actually about three significant categories, for **CCC** is equivalent to the category **RegσFra** of regular σ-frames and dually equivalent to **RegLinLoc**, the category of regular Lindelöf locales; see [MV1], [MV2] and [M1]. In **RegLinLoc**, the H-closed condition translates into the requirement that the subcategory should contain all the closed sublocales of any of its objects; see Lemma 1.3. The Main Theorem therefore implies that there is only one non-trivial closed-hereditary epi-coreflection in **RegLinLoc**. In **RegσFra**, the H-closed condition translates into the requirement that the subcategory should be closed with respect to "closed quotients"—a slightly weaker condition than H-closed; see the remarks after Lemma 1.5. The Main Theorem says that there is only one proper monoreflection in **RegσFra** that has this property; [MV2] implies that this is the boolean σ-frames.

Further interest comes from the analogy between rings of continuous real-valued functions and real-closed rings. The latter have been a subject of recent attention; see [Sc] and [SP]. Our Main Theorem is a precise analogue of Corollary 22.8 of [SM], which says that there is only one non-trivial H-closed monoreflection in the category of real-closed rings. It is remarkable that this result of [SM]—which appears somehow related to questions of first-order definability—has such a perfect analogue in **W**. **W** has no first-order axiomatization. The archimedean condition prevents this.

In much of the previous literature concerning **W**, **CCC** and **E**, the Yosida representation has been an important tool. The arguments in the present paper depend heavily on using methods from "pointless topology" [J1], especially the localic Yosida Theorem [MV2]. Using locales in place of spaces enables one to make certain limit constructions that do not have simple topological representations and to describe them in an efficient way that mirrors the intuitions that motivate them.

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1. Background and preliminaries

Reflections. Let $\mathbf{R}$ be a full subcategory of a category $\mathbf{C}$. $\mathbf{R}$ is said to be a *reflective subcategory* of $\mathbf{C}$ if it satisfies the following condition: for every object C in $\mathbf{C}$, there is an object ρC in $\mathbf{R}$ and a morphism $\rho_C : C \to \rho C$ with the property that for any morphism $f : C \to R$ with codomain in $\mathbf{R}$, there is a unique morphism $\overline{f} : \rho C \to R$ satisfying $\overline{f} \circ \rho_C = f$. We call ρC the *reflection of* C. If ρ_C always belongs to a class $\mathcal{K}$ of $\mathbf{C}$ morphisms, then $\mathbf{R}$ is said to be $\mathcal{K}$*-reflective.* In particular, if r_C is an epimorphism (*i.e.*, right-cancelable) for all $\mathbf{C}$ objects C, then $\mathbf{R}$ is said to be *epireflective.* Similarly, if r_C is always a monomorphism (*i.e.*, left-cancelable), then $\mathbf{R}$ is said to be *monoreflective.*

If $f : A \to B$ is a morphism of $\mathbf{C}$, then there is an induced morphism $\rho f := \overline{\rho_B \circ f} : \rho A \to \rho B$. Thus, ρ is a functor. We call it the *reflector* associated with $\mathbf{R}$. When we say ρ *is a* $\mathcal{K}$*-reflector*, we mean that ρ is the functor associated with a full $\mathcal{K}$-reflective subcategory.

Lemma 1.1. ([HS] 36.3). *Every full monoreflective subcategory is epireflective.* /////

Lemma 1.2. ([SM] 8.1). *Let* $\mathbf{S}$ *and* $\mathbf{R}$ *be full epireflective subcategories of* $\mathbf{C}$ *with reflectors* σ *and* ρ*, respectively. Then the following are equivalent:*

a) $\mathbf{S} \subseteq \mathbf{R}$

b) For every object C *of* $\mathbf{C}$*, there is a morphism* $\phi_C : \rho C \to \sigma C$ *such that* $\phi_C \rho_C = \sigma_C$*.*

c) There is a natural equivalence $\sigma \to \sigma\rho$ *having* $\sigma\rho_C$ *as its component at* C*.* /////

If $\mathbf{S} \subseteq \mathbf{R}$, we say $\mathbf{S}$ *is stronger than* $\mathbf{R}$. From the lemma it follows that if $\mathbf{S}$ is monoreflective and $\mathbf{S}$ is stronger than $\mathbf{R}$, then—by part b)—$\mathbf{R}$ is also a monoreflection, and by part c), ρC is a subobject of σC.

The categorical dual to a reflection is a coreflection and of course the duals of the definitions, remarks and observations above apply to coreflections.

Locales. A standard reference is [J2]. A frame is a complete lattice satisfying $a \wedge \bigvee B = \bigvee\{\, a \wedge b \mid b \in B \,\}$ and a morphism of frames is a function preserving all suprema and all finite infima. The category of locales, **Loc**, is the formal opposite of the category of frames. If $\mathcal{X}$ is a locale, $\tau(\mathcal{X})$ denotes the frame defining it. We call the elements of $\tau(\mathcal{X})$ "opens". If X is a topological space, $\tau(X)$ denotes its topology. $\mathcal{L}X$ denotes the locale corresponding to $\tau(X)$.

$\mathcal{P}$ denotes the one-point locale; $\tau(\mathcal{P}) = \{0, 1\}$. The *set of points of* $\mathcal{X}$ is the set of frame morphisms $p : \tau(\mathcal{X}) \to \tau(\mathcal{P})$ (equivalently, the set of **Loc** morphisms $\pi : \mathcal{P} \to \mathcal{X}$). The *space* of points of $\mathcal{X}$ is the same set endowed with the topology whose opens are the sets $\{\, p \mid p(a) = 1 \,\}$, $a \in \tau(\mathcal{X})$; see [J2], page 41. Depending on context, $\operatorname{pt}\mathcal{X}$ may denote the set of points or the space of points or even the locale associated with that space. We say $\operatorname{pt}\mathcal{X}$ is dense in $\mathcal{X}$ if every open in $\mathcal{X}$ contains a point, *i.e.*, for all non-zero $a \in \tau(\mathcal{X})$ there is a point p such that $p(a) = 1$. We say $\mathcal{X}$ is spatial if no two distinct opens contain the same points.

If X is completely determined by $\tau(X)$ (*i.e.*, X is sober; see [J2], II.1.6-7) then in many situations it is unnecessary to distinguish between X and $\mathcal{L}X$ (and when the distinction is unimportant and no confusion is possible, we sometimes omit the symbol $\mathcal{L}$). However, certain constructions (such as products and intersections) may yield different results depending whether they are carried out in **Top** or in **Loc**. Therefore, a notation that allows us to distinguish between categories is desirable. This is the main reason why we have the symbol $\mathcal{L}$.

Example. In general, $\mathcal{L}(\prod X_\lambda)$ (the locale determined by a topological product) is different from the localic product $\prod \mathcal{L}X_\lambda$. In particular, let $\mathbb{R}$ denote the space of real numbers and let $\mathcal{R} := \mathcal{L}\mathbb{R}$. If α is uncountable, then $\mathcal{L}(\mathbb{R}^\alpha) \neq \mathcal{R}^\alpha$. The latter is Lindelöf [DS], while the former is not [St]. On the other hand if E is at most countable, then $\mathcal{L}(\mathbb{R}^E) = \mathcal{R}^E$; see [DS]. If $\mathcal{X}_\varepsilon$ is a family of locales then $\operatorname{pt}\prod \mathcal{X}_\varepsilon$, the space of points of the localic product, is naturally homeomorphic with the product of spaces $\prod \operatorname{pt}\mathcal{X}_\varepsilon$. If for every index ε, $\operatorname{pt}\mathcal{X}_\varepsilon$ is dense in $\mathcal{X}_\varepsilon$, then $\operatorname{pt}\prod \mathcal{X}_\varepsilon$ is dense in the localic product $\prod \mathcal{X}_\varepsilon$.

RegLinLoc denotes the category of regular Lindelöf locales. The objects of this category admit several equivalent characterizations. They are the closed sublocales of the localic products $\mathcal{R}^\alpha$. Also, as discussed in more detail below, they are the locales corresponding to the frames of σ-ideals of a regular σ-frame. **RegLinLoc** is a monoreflective subcategory of the category of Tychonoff locales. The reflection of $\mathcal{X}$ is denoted $\lambda\mathcal{X}$. Note that the Hewitt realcompactification of a space X is $\operatorname{pt}\lambda X$, which is always dense in λX, even though λX often fails to be spatial. For proofs and discussion, see [MV1].

ℓ-groups. If $\mathcal{X}$ is any locale, $C(\mathcal{X})$ denotes the set of locale morphisms from $\mathcal{X}$ to $\mathcal{R}$ endowed with the ℓ-group structure induced by the operations in $\mathcal{R}$; see [M1], section 2. We view $C(\mathcal{X})$ as an object of **W** by letting the constant function 1 play the role of weak unit. Note that $C(X)$, the ring of real-valued continuous functions on the space X, is naturally isomorphic to $C(\mathcal{L}X)$. It is also the case that $C(\mathbb{R}^\alpha)$ is naturally isomorphic to $C(\mathcal{R}^\alpha)$. This follows from lemma 5.1 of [MV1] and the facts cited in the example in the previous subsection. (Lemma 5.1 seems to have some unnecessary hypotheses; a more general version is implicit in Lemma 4.2, below.)

By a theorem of Isbell, an archimedean ℓ-group with weak order unit is isomorphic to some $C(\mathcal{X})$ iff it is closed under countable composition; see [MV2]. The full subcategory of **W** whose objects are those ℓ-groups isomorphic to some $C(\mathcal{X})$ is denoted **CCC**, the name being a reminder of the distinctive property its objects enjoy. As mentioned in the introduction, **CCC** is monoreflective in **W** and **CCC** contains **E**, which is also monoreflective in **W**. The reflectors are denoted γ and β, respectively. Also, **E** being the strongest monoreflective subcategory of **W**, we deduce from lemma 1.2 that for any monoreflection μ of **W** and any A in **W**, μA is isomorphic to a sub-ℓ-group of βA. (Monomorphisms in **W** are injective.)

If A is any **W** object, its Yosida locale, denoted $\mathcal{Y}(A)$ is the locale determined by the frame consisting of those relatively-uniformly closed ideals of A that are contained within the ideal generated by the weak unit; see [M1]. $\mathcal{Y}(A)$ is regular Lindelöf. There is a canonical embedding $a \mapsto \hat{a} : A \to C(\mathcal{Y}(A))$ called the "localic Yosida representation." In fact, the functors $\mathcal{Y}$ and C form an adjoint pair that give rise to the reflections mentioned earlier: $\mathcal{Y}(C(\mathcal{X})) = \lambda\mathcal{X}$ and $C(\mathcal{Y}(A)) = \gamma A$. These functors provide a dual equivalence between **RegLinLoc** and **CCC**.

Lemma 1.3. *Suppose $\mathcal{X}$ is regular Lindelöf. Let $\phi : C(\mathcal{X}) \to B$ be a surjective* **W***-morphism. Then B is naturally isomorphic to $C(\mathcal{Z})$, where $\mathcal{Z}$ is a closed sublocale of $\mathcal{X}$.*

Conversely, given any closed sublocale $\mathcal{Z}$ in $\mathcal{X}$, restriction to $\mathcal{Z}$ produces a surjective **W** *morphism $C(\mathcal{X}) \to C(\mathcal{Z})$.*

Proof sketch. The kernel K of ϕ is a relatively-uniformly closed ideal of $C(\mathcal{X})$–hence an element of $\tau(\mathcal{X})$. $U \mapsto U \vee K$ is a "closed" nucleus in $\tau(\mathcal{X})$. This determines the sublocale $\mathcal{Z}$. The localic Tietze Extension Theorem (see, *e.g.*, [LW]) shows that every function in $C(\mathcal{Z})$ is the restriction to $\mathcal{Z}$ of some element of $C(\mathcal{X})$. /////

The objects $C(\mathcal{R}^E)$ play a special role. Let E be a set and let $\pi_\varepsilon : \mathcal{R}^E \to \mathcal{R}$ denote the canonical projection onto the ε^{th} factor.

Lemma 1.4. *As an object of* **CCC**, *$C(\mathcal{R}^E)$ is freely generated by $\{\, \pi_\varepsilon \mid \varepsilon \in E \,\}$.*

Proof. Let $\{\, g_\varepsilon \mid \varepsilon \in E \,\}$ be a set of elements of $C(\mathcal{X})$. Then by the definition of the product, there is a unique morphism $g : \mathcal{X} \to \mathcal{R}^E$ such that $\pi_\varepsilon \circ g = g_\varepsilon$. The function $g^\sharp := C(g)$ from $C(\mathcal{R}^E)$ to $C(\mathcal{X})$ satisfies $g^\sharp(\pi_\varepsilon) = g_\varepsilon$ and is the unique function with this property. /////

It follows that each **CCC** object is a homomorphic image of some $C(\mathcal{R}^E)$. In light of Lemma 1.3, this is dual to the fact that each regular Lindelöf locale is a closed sublocale of some $\mathcal{R}^E$.

σ-frames. A σ-frame is a lattice S with least element 0 and greatest element 1 in which every finite or countable set $Y \subseteq S$ has a supremum $\bigvee Y$ and in which the following countable distributive law holds: $x \wedge \bigvee Y = \bigvee\{\, x \wedge y \mid y \in Y \,\}$. S is said to be *regular* if for each $x \in S$, there is are sets $\{y_i\}_{i=1}^{\infty}$ and $\{z_i\}_{i=1}^{\infty}$ in S such that $x = \bigvee_{i=1}^{\infty} y_i$, and for each i, $y_i \wedge z_i = 0$ and $x \vee z_i = 1$. $I \subset S$ is a σ-ideal if $x \le y \in I$ implies $x \in I$ and I contains the supremum of any finite or countable subset of I.

For any σ-frame S, the collection of all σ-ideals of S forms a frame. (Meet is intersection; the join of a set of σ-ideals is the σ-ideal generated by their union.) $\mathcal{Y}_\sigma(S)$ denotes the locale determined by this frame. Conversely, for any locale $\mathcal{X}$, the set of all cozero elements of $\mathcal{X}$, denoted $\mathbf{coz}(\mathcal{X})$, is a regular σ-frame. $\mathcal{Y}_\sigma$ and **coz** are in fact functors, and in analogy with $\mathcal{Y}$ and C, they form an adjoint pair. $\mathcal{Y}_\sigma(\mathbf{coz}(\mathcal{X})) = \lambda\mathcal{X}$, while $\mathbf{coz}(\mathcal{Y}_\sigma(S))$ is the regularization of S. These functors restrict to a dual equivalence between **RegLinLoc** and **RegσFra**; see [MV2] and [M2].

S is *boolean* if each element has a complement, *i.e.*, for each $y \in S$ there is $y' \in S$ such that $y \vee y' = 1$ and $y \wedge y' = 0$. Every boolean σ-frame is regular. (Proof: Given x, let $y_i \equiv x$ and $z_i \equiv x'$.) The full subcategrory of **RegσFra** consisting of the boolean σ-frames is monoreflective. The boolean reflection of S, denoted βS, may be constructed as follows. Define S' by freely adjoining a complement z' for every element $z \in S$. The elements of S' can all be expressed in the form $\bigvee_{i=1}^{\infty}(y_i \wedge z_i')$, but such elements may not be complemented in S'. If we iterate this operation up to the first uncountable ordinal, however, the result is βS. This construction is detailed in [M2], where plenty of additional information about σ-frames is provided. (As a referee has pointed out, the Boolean reflection of S can also be constructed by taking the quotient of the free boolean σ-algebra on the underlying set of S modulo the relations of S. Our description makes explicit the existence of a hierarchy of sub-σ-frames of βS that is reminiscent of descriptive set theory.)

Lemma 1.5. ([MV2]). *$C(\mathcal{X}) \in \mathbf{E}$ if and only if $\mathbf{coz}(\mathcal{X})$ is boolean.* /////

A σ-frame congruence on a σ-frame S is a subset of $S \times S$ that is both an equivalence relation on S and a sub-σ-frame of $S \times S$. The set of equivalence classes of any congruence is naturally a σ-frame. If a congruence Θ is generated by a set of the form $\{ (0, y) \mid y \in Y \}$ for some $Y \subseteq S$, then we say that Θ is *closed*. The closed congruences on $\mathbf{coz}(\mathcal{X})$, $\mathcal{X}$ regular Lindelöf, are exactly those congruence obtained by restricting cozero elements to a closed sublocale $\mathcal{Z}$ of $\mathcal{X}$. (The cozero elements that do not meet $\mathcal{Z}$ are identified with 0.) We say that a class of σ-frames is herediary with respect to closed congruences if whenever S is in the class and Θ is a closed congruence on S, then S/Θ is also in the class. For detailed information about congruences, see [M2].

2. Further facts related to points

Let b be the coreflector in **RegLinLoc** corresponding to epicompletion β in **CCC** and let $\mathcal{X}$ be a regular Lindelöf locale. (Thus, $\beta C(\mathcal{X}) = C(b\mathcal{X})$.) It is seldom the case that $b\mathcal{X}$ is spatial, even when $\mathcal{X}$ is. For example, as a space $\mathrm{pt}\, b\mathcal{R}$ is discrete, but (as we shall elaborate in a moment) $\beta C(\mathcal{R}) \neq \mathbb{R}^{\mathbb{R}}$. This means that $b\mathcal{R} \neq \mathcal{L}\,\mathrm{pt}\, b\mathcal{R}$, *i.e.*, $b\mathcal{R}$ is not spatial. In fact, $\mathrm{pt}\, b\mathcal{X}$ *may even fail to be dense* in $b\mathcal{X}$, though $\mathcal{X}$ be spatial. $\mathcal{X} = \mathcal{L}\mathbb{Q}$ is an example; see [MV2]. A significant part of the proof of the Main Theorem depends upon the analysis of the behavior of a certain element of $C(b\mathcal{R})$ near a point of $b\mathcal{R}$. As the phenomena just cited suggest, the relationships between the points of $b\mathcal{X}$ and the elements of $C(bX)$ are subtle. In this section, we develop the needed facts.

It is well-known that topological coreflections do not change the points of a space but merely add open sets. For locales there is an analogous fact:

Lemma 2.1. *Suppose $\mathbf{C}$ is any full subcategory of $\mathbf{Loc}$ and $\mathbf{D}$ is a full coreflective subcategory of $\mathbf{C}$ with coreflector d. Suppose also that the one-point locale $\mathcal{P}$ is in $\mathbf{D}$. Then for each $\mathcal{X}$ in $\mathbf{C}$, $\mathrm{pt}\,\mathcal{X} = \mathrm{pt}\, d\mathcal{X}$ as sets.*

Proof. Let $\pi : \mathcal{P} \to d\mathcal{X}$ be a point of $d\mathcal{X}$. The composition of π with $d_{\mathcal{X}} : d\mathcal{X} \to \mathcal{X}$ gives a point of $\mathcal{X}$. Conversely, given any $\pi : \mathcal{P} \to \mathcal{X}$, there is unique $\overline{\pi} : \mathcal{P} \to d\mathcal{X}$ such that $d_{\mathcal{X}} \circ \overline{\pi} = \pi$. The maps $\pi \mapsto d_{\mathcal{X}} \circ \pi$ and $\pi \mapsto \overline{\pi}$ are clearly inverses of one another. /////

Recall that the algebra of Baire sets $\mathbf{ba}(X)$ in a space X is the σ-complete boolean algebra generated in the power set of X by the cozero sets. Let $B(X)$ denote the ℓ-group of Baire functions on X, *i.e.*, the functions $f : X \to \mathbb{R}$ such that $f^{-1}((a, b))$ is Baire for all open intervals (a, b). In [BH], the authors show that $B(X)$ is epicomplete. Let X be a Tychonoff space and let $\phi : C(X) \to B(X)$ be the natural inclusion. The induced morphism $\overline{\phi} : \beta C(X) \to B(X)$ is a surjection; see 2.3 of [B1]. The example of $\mathcal{X} = \mathcal{L}\mathbb{Q}$ shows that $\overline{\phi}$ may fail to be an injection. The problem of when $\overline{\phi}$ *is* injective is considered in [B1]. The following lemma is an elaboration of Proposition 2.4 of that paper:

Lemma 2.2. *Let X be a Tychonoff space. The following are equivalent:*

1) *$\overline{\phi} : \beta C(X) \to B(X)$ is injective. (Note that β here refers to the epicompletion.)*
2) *The natural map $\mathbf{coz}(b\lambda\mathcal{L}X) \to \mathbf{ba}(X)$ is injective.*
3) *Each Baire set of βX that misses X is contained in a countable union of zero-sets of βX that misses X. (Note that β here refers to the Stone-Čech compactification.)*

Proof. The equivalence of 1) and 3) is proved in the reference cited. The equivalence of 2) and 3) follows by a similar argument, modified in the way exemplified in [MV2], 5.1./////

Those spaces for which the equivalent conditions of 2.2 hold are called ε-spaces. The information relevant to us is Theorem 4.1 of [B1], which says that if X is Lindelöf and Čech-complete, then it is an ε-space. *In particular, if E is finite or countable, then the epicompletion $\beta C(\mathbb{R}^E)$ is isomorphic to the ℓ-group of Baire functions $B(\mathbb{R}^E)$.*

Lemma 2.3. *Let μ be a monoreflector in* **CCC** *and let m be the corresponding coreflector in* **RegLinLoc**. *Let E be a finite or countable set. Then*

a) $\mu C(\mathbb{R}^E) \subseteq B(\mathbb{R}^E) \subseteq \mathbb{R}^{\mathbb{R}^E}$ and $\mathbf{coz}\big(m(\mathbb{R}^E)\big)$ is a sub-σ-frame of $\mathbf{ba}(\mathbb{R}^E)$.

b) $\mu C(\mathbb{R}^E)$, viewed as a subset of $\mathbb{R}^{\mathbb{R}^E}$, is closed under composition. In other words, suppose f and $\{\, g_\varepsilon \mid \varepsilon \in E \,\}$ belong to the natural image of $\mu C(\mathbb{R}^E)$ in $\mathbb{R}^{\mathbb{R}^E}$. Let $g : \mathbb{R}^E \to \mathbb{R}^E$ be the map induced by the g_ε. Then $f \circ g$ belongs to the natural image of $\mu C(\mathbb{R}^E)$ in $\mathbb{R}^{\mathbb{R}^E}$.

Proof. The first assertion in a) follows from the remarks just before the lemma and the remarks following Lemma 1.2. For the second assertion, recall that the reflection map from regular σ-frames to Boolean σ-algebras is injective (and not merely monic). Regarding part b), note that $m(\mathbb{R}^E)$ need not be a spatial locale (but nonetheless the points are dense). For the time being, we will view the functions f and g_ε as locale morphisms rather than as point functions. The $g_\varepsilon : m(\mathbb{R}^E) \to \mathbb{R}$ induce a morphism $g : m(\mathbb{R}^E) \to \mathbb{R}^E$, and by the universal property of m, this lifts to a morphism $\overline{g} : m(\mathbb{R}^E) \to m(\mathbb{R}^E)$. The composition $f \circ \overline{g}$ belongs to $C\big(m(\mathbb{R}^E)\big)$. If we now restrict to the points of $m(\mathbb{R}^E)$, we get the desired result. /////

In the proof of the main theorem, below, we need only a special case of the last two lemmas, namely that every cozero set of $m\mathbb{R}$ is a Baire set in $\mathbb{R}$ and that $\mu C(\mathbb{R})$ is a sub-ℓ-group of $\mathbb{R}^{\mathbb{R}}$ that contains $C(\mathbb{R})$ and is closed under composition. In the next section, we examine an elementary consequence of the latter hypothesis.

3. Simplifying a Discontinuity

The following result is the key geometric idea in the proof of the main theorem. Since it is an elementary result that does not depend on locales, we present it separately.

Proposition 3.1. *Suppose that A is a sub-ℓ-group of $\mathbb{R}^{\mathbb{R}}$ that contains $C(\mathbb{R})$ and is closed under composition. If A contains a function that is not continuous, then it contains the function*

$$\delta(x) := \begin{cases} 1, & \text{if } x = 0; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Suppose that $f \in A$ and f is not continuous. By changing coordinates—*i.e.*, by composing with suitable translations—we may assume that f is not continuous at 0 and that $f(0) = 0$. Since $f = (f \vee 0) - (-f \vee 0)$, either $f \vee 0$ or $-f \vee 0$ must fail to be continuous at 0. There is no loss of generality, therefore, in assuming that $f \geq 0$. We may also assume that f is not continuous from the right at 0, by replacing $f(x)$ with $f(-x)$ if necessary. Then there is $\epsilon > 0$ with the property that every interval $(0, \gamma)$ contains some x such that $f(x) > \epsilon$. Let $g := \frac{1}{\epsilon}(f \wedge \epsilon)$. Then, $0 \leq g \leq 1$, $g(0) = 0$ and every interval $(0, \gamma)$ contains some x such that $g(x) = 1$. Let $x_1 > x_2 > \cdots$ be a decreasing sequence converging to 0, with $g(x_i) = 1$ for all $i = 1, 2 \ldots$. We now define two continuous functions $h, k : \mathbb{R} \to \mathbb{R}$. For $n = 1, 2, \ldots$, we set

$$k\left(\frac{\pm 1}{2n+1}\right) = x_n\ , \quad k\left(\frac{\pm 1}{2n}\right) := x_n\ , \quad h\left(\frac{\pm 1}{2n}\right) := x_n \text{ and } h\left(\frac{\pm 1}{2n-1}\right) := x_n\ .$$

We set $h(x) = x_1$ if $|x| > 1$, and $k(x) = x_1$ if $|x| > 1/2$. The values of h and k in the interiors of intervals between consecutive numbers of the form $\pm\frac{1}{m}$ $(m = 1, 2, \ldots)$ are to be determined by linear interpolation. Finally, we set $h(0) = k(0) = 0$. Because $x_i \to 0$, both h and k are continuous at 0. From the way we have defined h and k, they are also continuous at all other points. Now

- $0 \leq g \circ h \leq 1$,
- $g \circ h(0) = 0$,
- $g \circ h \equiv 1$ outside $(-1, 1)$, and
- $g \circ h \equiv 1$ on each interval $\left[\frac{1}{2n}, \frac{1}{2n-1}\right]$ and its negative $\left[\frac{-1}{2n-1}, \frac{-1}{2n}\right]$, $n = 1, 2, \ldots$.

Similarly,

- $0 \leq g \circ k \leq 1$,
- $g \circ k(0) = 0$,
- $g \circ k \equiv 1$ outside $(-\frac{1}{2}, \frac{1}{2})$, and
- $g \circ k \equiv 1$ on each interval $\left[\frac{1}{2n+1}, \frac{1}{2n}\right]$ and its negative $\left[\frac{-1}{2n}, \frac{-1}{2n+1}\right]$, $n = 1, 2, \ldots$.

Thus $\delta = 1 - \big((g \circ h) \vee (g \circ k)\big) \in A$. /////

4. Proof of Main Theorem.

Let $\mathbf{M}$ be a proper H-closed monoreflection in $\mathbf{CCC}$, with reflector μ, and let $m_{\mathcal{X}} : m\mathcal{X} \to \mathcal{X}$ denote the co-reflection in **RegLinLoc** corresponding to μ, so that $\mu C(\mathcal{R}^\alpha) = C\big(m(\mathcal{R}^\alpha)\big)$. Our goal is to show that $\mathbf{M} = \mathbf{E}$. The strategy is as follows. First note that if ν is any other H-closed monoreflection of $\mathbf{CCC}$, then for some cardinal ξ, $\mu C(\mathcal{R}^\xi) \neq \nu C(\mathcal{R}^\xi)$; see lemma 1.4 and the remark following it. Pick α so that $\mu C(\mathcal{R}^\alpha) \neq C(\mathcal{R}^\alpha)$. Our first step is to show (in 4.1) that $C\big(m(\mathcal{R}^\alpha)\big)$ contains a function that is discontinuous at some point of $\mathbb{R}^\alpha$. Second, using the description of the boolean reflection in regular σ-frames, we show in 4.2 that we can assume that α is countable. In 4.3, using the H-closed condition, we show that we can actually assume $\alpha = 1$. Now, Proposition 3.1 implies that $\mu C(\mathcal{R})$ (viewed as a sub-ℓ-group of $\mathbb{R}^\mathbb{R}$) contains the function δ. From this, we deduce (4.4) that for every $\mathcal{X}$, $\mathbf{coz}(m\mathcal{X})$ is boolean, and by Lemma 1.5 we are done.

The details of this argument are provided in the following lemmas.

Lemma 4.1. *$C\big(m(\mathcal{R}^\alpha)\big)$ contains a function—henceforth called Δ—that is discontinuous at some point of $\mathbb{R}^\alpha$ with respect to the standard topology on $\mathbb{R}^\alpha$.*

Proof. Restriction to $\mathbb{R}^\alpha = \operatorname{pt} m(\mathcal{R}^\alpha) \subset m(\mathcal{R}^\alpha)$ induces a $\mathbf{W}$-morphism from $C\big(m(\mathcal{R}^\alpha)\big)$ to $\mathbb{R}^{\mathbb{R}^\alpha}$. Since $\mathbf{M}$ is H-closed, the image belongs to $\mathbf{M}$. Moreover, this image, which is a set of real-valued functions on $\mathbb{R}^\alpha$, contains all those functions that are continuous with respect to the standard topology on $\mathbb{R}^\alpha$. But by assumption $C(\mathbb{R}^\alpha)$ is not in $\mathbf{M}$. Thus, the image must contain a function that is not continuous and this obviously must fail to be continuous at some point. /////

Lemma 4.2. *Let $f \in C\big(m(\mathcal{R}^\alpha)\big)$. There is a countable set $E \subseteq \alpha$ such that f factors through the reflection of the projection $m(\pi) : m(\mathcal{R}^\alpha) \to m(\mathcal{R}^E)$, i.e., $f = f' \circ m(\pi)$ for some f'.*

Proof. It is most convenient to argue in terms of the σ-frame reflection corresponding to μ. We will use the symbol μ_σ to denote the reflector. Then,

$$\mu C(\mathcal{R}^\alpha) = \operatorname{Mor}(\mathbf{coz}(\mathcal{R}), \mu_\sigma \, \mathbf{coz}(\mathcal{R}^\alpha)).$$

Now any $f : \mathbf{coz}(\mathcal{R}) \to \mu_\sigma \, \mathbf{coz}(\mathcal{R}^\alpha)$ is completely determined by the images $f\big((r,s)\big)$ of the rational intervals (r,s), where $r, s \in Q$ and $r < s$. Of these, there are countably many. Additionally, $\mu_\sigma \, \mathbf{coz}(\mathcal{R}^\alpha)$ is contained in the boolean reflection of $\mathbf{coz}(\mathcal{R}^\alpha)$ (by Lemma 1.2.), and every element of the boolean reflection of *any* sigma-frame S can be expressed in terms of countably many elements of S. Finally, $\mathbf{coz}(\mathcal{R}^\alpha)$ is a homomorphic image of the σ-frame coproduct $\Sigma_\alpha \, \mathbf{coz}(\mathcal{R})$. Each element of this is a countable supremum of elements, each a finite infimum of the form $J_1 \wedge J_2 \wedge \ldots J_n$, where each J_i is an open sub-interval of one of the summands. We see, then, that each $f\big((r,s)\big)$ can be expressed in terms of countably many coordinates, and f being determined by countably many such expressions, the lemma follows. /////

We may now assume that $\alpha \leq \omega$.

Lemma 4.3. *$\mu C(\mathcal{R})$ contains a function that is discontinuous at some point of $\mathbb{R}$ with respect to the standard topology on $\mathbb{R}$.*

Proof. Let b_∞ be a point of discontinuity of the function $\Delta : \mathbb{R}^\omega \to \mathbb{R}$ selected previously. There is a sequence $\{ b_i \mid i = 1, 2, \dots \}$ converging to b such that $\Delta(b_\infty) \neq \lim_{i\to\infty} \Delta(b_i)$. Define $g : \mathbb{R} \to \mathbb{R}^\omega$ as follows: let $g(x) = b_\infty$ if $x \leq 0$; let $g(1/n) = b_n$ for $n = 2, 3, \cdots$; let $g(x) = b_1$ for $x \geq 1$; and let g be linear on each interval $[\frac{1}{n}, \frac{1}{n+1}]$, $n = 1, 2, \dots$. Applying m, we get a continuous morphism $mg : m\mathbb{R} \to m(\mathbb{R}^\omega)$ and hence a **W**-morphism $C(mg) : C\big(m(\mathbb{R}^\omega)\big) \to C(m\mathbb{R})$. The restriction of $C(mg)(\Delta)$ to $\mathbb{R} = \mathrm{pt}\, m\mathbb{R}$ is just $\Delta \circ g$, and this is clearly discontinuous. /////

We draw some immediate conclusions from 4.3. By 3.1, $\mu C(\mathcal{R})$, viewed as a subset of $\mathbb{R}^\mathbb{R}$, contains δ. By 2.3.b, if $f \in \mu C(\mathbb{R})$, then $\delta \circ f \in \mu C(\mathbb{R})$. By 2.3.a, $\mathrm{coz}(f)$ is the complement of $\mathrm{coz}(\delta \circ f)$ in $\mathbf{coz}(m\mathcal{R})$. Thus, we see already that $\mathbf{coz}(m\mathcal{R})$ is boolean. Our last lemma extends this result. In the proof, we make reference (for the first time in this paper) to the cozero element $\mathrm{coz}(f)$ determined by a specific element $f \in C(\mathcal{X})$. Note that $\mathrm{coz}(|f| \vee |g|) = \mathrm{coz}(f) \vee \mathrm{coz}(g)$ and $\mathrm{coz}(|f| \wedge |g|) = \mathrm{coz}(f) \wedge \mathrm{coz}(g)$ in $\mathbf{coz}(\mathcal{X})$. Also note that if $\phi : C(\mathcal{X}) \to C(\mathcal{Y})$ is a **W**-morphism and $\phi_\sigma : \mathbf{coz}(\mathcal{X}) \to \mathbf{coz}(\mathcal{Y})$ is the induced morphism of σ-frames, then $\mathrm{coz}(\phi(f)) = \phi_\sigma(\mathrm{coz}(f))$.

Lemma 4.4. *For any regular Lindelöf locale $\mathcal{X}$, $\mathbf{coz}(m\mathcal{X})$ is boolean.*

Proof. Let $\mu_{C(\mathcal{R})} : C(\mathcal{R}) \to \mu C(\mathcal{R})$ denote the reflection morphism. Suppose $g \in C(m\mathcal{X})$. There is a morphism $\phi : C(\mathcal{R}) \to C(m\mathcal{X})$ such that $\phi(\mathrm{id}_\mathcal{R}) = g$. This extends to a map $\overline{\phi} : \mu C(\mathcal{R}) \to C(\mathcal{X})$. Let $h := \mu_{C(\mathcal{R})}(\mathrm{id}_\mathcal{R})$. Then observe that

$$\begin{aligned}\mathrm{coz}(g) \vee \mathrm{coz}(\overline{\phi}(\delta)) &= \mathrm{coz}\big(|\overline{\phi}(h)| \vee \overline{\phi}(\delta)\big)\\ &= \mathrm{coz}\big(\overline{\phi}(|h| \vee \delta)\big)\\ &= \phi_\sigma\big(\mathrm{coz}(h) \vee \mathrm{coz}(\delta)\big) = 1.\end{aligned}$$

(We justify the last line by lemma 2.3.a.) A similar argument shows $\mathrm{coz}(g) \wedge \mathrm{coz}(\overline{\phi}(\delta)) = 0$. /////

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