On infinitely near points

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`http://www.math.lsu.edu/~madden`

The goal of the work reported here is to develop a computational approach to linear systems defined by infinitely near base conditions. The approach that I am developing depends on concrete computations with actual polynomials and forces one to explore complicated combinatorial structures in polynomial rings. The results I can report, insofar as they have geometric meaning, refer to affine space. I hope, however, that insights obtained in the manner I am proceeding might lead to generalizations.

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Certain informal notational conventions will be used in these notes. We generally use X to refer to an affine space of dimension d, while W is used to denote a space obtained by blowing up a point in some other space. We often deal with a sequence of blow-ups. Superscripts in parentheses—as in $W^{(i)}$—keep track of how many blow-ups have been performed. We also use such superscripts to mark symbols that refer to subsets of $W^{(i)}$ or to coordinate functions in $W^{(i)}$. In instances when an object is associated to several different $W^{(i)}$'s, the superscript reflects the $W^{(i)}$ most relevant in context. All definitions can be understood without reference to this convention, which is intended to serve as a reminder only.

1. Point sequences.

The following definition introduces the subject of these notes.

Definition. *Let k be a field and let $X^{(0)}$ be affine d-space over k. Let $\pi_1 : W^{(1)} \to X^{(0)}$ be the blow-up of the origin $O_0 \in X^{(0)}$, let $\pi_2 : W^{(2)} \to W^{(1)}$ be the blow-up of a k-rational point $O_1 \in E_0 := \pi_1^{-1}(O_0) \subseteq W^{(1)}$. Suppose this is continued for n steps. We get a sequence of blow ups*

$$W^{(n)} \xrightarrow{\pi_n} \cdots \xrightarrow{\pi_3} W^{(2)} \xrightarrow{\pi_2} W^{(1)} \xrightarrow{\pi_1} X^{(0)},$$

where

$$\pi_{i+1} \text{ is the blow-up of } O_i \in E_{i-1} := \pi_i^{-1}(O_{i-1}) \subseteq W^{(i)}.$$

A sequence of points $\{O_i\}$ meeting these conditions will be called a k-rational point sequence. (One might consider points with coordinates not in necessarily in k but only in an extension of k. We do not need this level of generality here, however.)

For $j > i+1$, let $E_i^{(j)} \subset W^{(j)}$ denote the proper transform of $E_i \subset W^{(i+1)}$ under the morphism $\pi_j \circ \ldots \circ \pi_{i+2}$. It is consistent with this convention to let $E_i^{(i+1)}$ denote E_i, and we shall use this notation occasionally.

Definition. *We say that O_j is proximate to O_i (in symbols, $O_j \to O_i$) if $O_j \in E_i^{(j)}$.*

Lemma 1.1. *The proximity relation on any closed point sequence satisfies the following conditions:*

i) for all i and j, $O_j \to O_i$ implies $j > i$,

ii) for all $i \geq 0$, $O_{i+1} \to O_i$, and

iii) for all $j > i+1 \geq 1$, $O_j \to O_i$ implies $O_{j-1} \to O_i$.

Proof. These follow immediately from the definitions. /////

2. Coordinate tables.

Let $\{O_i\}$ be a k-rational point sequence. We shall describe a canonical way to choose affine spaces $X^{(i)} \subseteq W^{(i)}$ and local coordinates in $X^{(i)}$ with O_i at the origin. We begin with given coordinates $x_1^{(0)}, \ldots, x_d^{(0)}$ in $X^{(0)}$ with O_0 at the origin. Let

$$\mathcal{A}^{(0)} := k[x_1^{(0)}, \ldots, x_d^{(0)}]$$

be the (k-rational) coordinate ring of $X^{(0)}$.

For convenience, we will temporarily suppress the superscripts on the $x_i^{(0)}$, writing simply x_i instead. If $W^{(1)}$ is the result of blowing up O_1, we may write $W^{(1)}$ as a union of overlapping affine d-spaces:

$$W^{(1)} = X_1^{(1)} \cup \ldots \cup X_d^{(1)}$$

Here $X_j^{(1)}$ denotes the affine subset of $W^{(1)}$ on which the functions

$$\frac{x_1}{x_j}, \ \ldots, \ \frac{x_{j-1}}{x_j}, \, x_j, \, \frac{x_{j+1}}{x_j}, \ \ldots, \ \frac{x_d}{x_j} \tag{2.0}$$

are well-defined everywhere. Note that $x_j = 0$ is the local equation of the exceptional divisor E_0 within $X_j^{(1)}$. (The facts quoted in this paragraph are more fully explained in any standard reference on blowing up.)

We now construct $X^{(1)}$ and its coordinate system. Let $j(1)$ be the least of the integers $j = 1, \ldots, d$ for which $O_1 \in X_j^{(1)}$ and set $X^{(1)} := X_{j(1)}^{(1)}$. Consider the coordinates of O_1 in the coordinate system (2.0), with $j = j(1)$. Observe that 0 *must* occur in position $j(1)$ because by assumption O_1 is in E_0. Observe further that when $j < j(1)$, not all the rational functions in (2.0) are defined at O_1, since we have assumed that $j(1)$ is the least index for which they *are* defined. Thus in $X^{(1)}$, the coordinates of O_1 (with respect to (2.0)) look like: $(0, \ldots, 0, a_{j(1)+1}^{(1)} \ldots, a_d^{(1)})$, where the $a_i^{(1)}$ may be any elements of k.

Set

$$x_j^{(1)} := \begin{cases} x_j/x_{j(1)}, & \text{if } j < j(1); \\ x_{j(1)}, & \text{if } j = j(1); \\ (x_j/x_{j(1)}) - a_j^{(1)}, & \text{if } j > j(1). \end{cases}$$

Now, the functions $x_1^{(1)}, \ldots, x_d^{(1)}$ give a coordinate system on $X^{(1)}$ with O_1 at the origin. Let $\mathcal{A}^{(1)} := k[x_1^{(1)}, \ldots, x_d^{(1)}]$. Writing x' in place of $x^{(1)}$, our definitions imply:

$$x_j := \begin{cases} {x'}_{j(1)} {x'}_j, & \text{if } j < j(1); \\ {x'}_{j(1)}, & \text{if } j = j(1); \\ {x'}_{j(1)} \big({x'}_j + {a'}_j\big), & \text{if } j > j(1), \end{cases}$$

which gives the homomorphism from $\mathcal{A}^{(0)}$ to $\mathcal{A}^{(1)}$ explicitly.

From this point, we can proceed exactly as we did in passing from $\mathcal{A}^{(0)}$ to $\mathcal{A}^{(1)}$ to make $\mathcal{A}^{(2)}$. Continuing, we construct a sequence of ring embeddings

$$\mathcal{A}^{(0)} \to \mathcal{A}^{(1)} \to \mathcal{A}^{(2)} \to \cdots.$$

The data that we thus develop is uniquely determined by the point sequence $\{O_i\}$ and the original coordinate system chosen for $X^{(0)}$.

We introduce one additional bit of useful notation. Let

$$\varepsilon_n := x^{(i+1)}_{j(i+1)} \in \mathcal{A}^{(i+1)}. \qquad 2.1$$

Then, $\varepsilon_i = 0$ is the local equation for the exceptional divisor E_i in $X^{(i+1)}$.

Coordinate tables provide a convenient way of recording the data that determines the sequence $\{\mathcal{A}^{(i)}\}$. Here is how to construct them. If $i \neq j(i)$, then we have specified how to choose $a^{(i)}_j \in k$, above. (We take $a^{(i)}_j = 0$ when $i < j(i)$.) We have not specified a meaning for $a^{(i)}_{j(i)}$, however. We shall simply set

$$a^{(i)}_{j(i)} := * , \qquad 2.2$$

where $*$ is a symbol not in k. The asterisk is nothing more than a notational device whose position indicates the value of $j(i)$.

The matrix $\{a^{(i)}_j\}$ is called a *coordinate table* for $\{O_i\}$. We shall write these tables with the i-index constant along rows, *e.g.*,

$$\begin{array}{cccc|c}
- & - & - & - & O_0 \\
0 & 0 & * & a^{(1)}_4 & O_1 \\
0 & * & a^{(2)}_3 & a^{(2)}_4 & O_2 \\
* & a^{(3)}_2 & a^{(3)}_3 & a^{(3)}_4 & O_3 \\
0 & 0 & * & a^{(4)}_4 & O_4
\end{array} \; .$$

Here, we have drawn a coordinate table with an extra column appended on the right, in which we name the point whose coordinates appear in the same row.

The proximity relations of $\{O_i\}$ may be determined from the position of the asterisks and zero entries in any coordinate table, as the following shows.

Lemma 2.1. *Let $\{a^{(i)}_j\}$ be a coordinate table of a k-rational point sequence $\{O_i\}$. Then $O_n \to O_s$ if and only if $n = s+1$ or $n > s+1$ and $0 = a^{(s+2)}_{j(s+1)} = \ldots = a^{(n)}_{j(s+1)}$.*

Proof. Since $\{Q_i = O_{s+i}\}$ is a k-rational point sequence, it is enough to prove the lemma under the assumption that $s = 0$. If $0 = a^{(2)}_{j(1)} = \ldots = a^{(n)}_{j(1)}$, then $j(1)$ does not occur among the indices $j(2), \ldots, j(n)$. The vanishing of these constants also shows that $E^{(i)}_0 \cap X^{(i)} = V(x^{(i)}_{j(1)})$ for $i = 1, \ldots, n$. Since O_n is determined by $x^{(n)}_1 = \ldots = x^{(n)}_d = 0$, we see $O_n \in E^{(n)}_0$, *i.e.*, $O_n \to O_0$. To prove the other implication, we use induction. The

$n = 1$ case is obvious. Suppose $n > 1$ and $O_n \to O_0$. By Lemma 1.1.*iii*, $O_{n-1} \to O_0$ so by induction $0 = a^{(2)}_{j(1)} = \ldots = a^{(n-1)}_{j(1)}$. It follows that $E^{(n-1)}_0 \cap X^{(n-1)} = V(x^{(n-1)}_{j(1)})$. Now $j(n) = j(1)$ is impossible, because this implies $E^{(n)}_0 \cap X^{(n)} = \emptyset$, but we assumed $O_n \in E^{(n)}_0 \cap X^{(n)}$. Thus, $E^{(n)}_0 \cap X^{(n)} = V(x^{(n)}_{j(1)} + a^{(n)}_{j(1)})$. Since by hypothesis O_n belongs to this set, $a^{(n)}_{j(1)} = 0$. /////

For example, suppose the following coordinate table is given:

$$
\begin{array}{ccccc|l}
- & - & - & - & & O_0 \\
0 & 0 & * & 1 & & O_1 \\
0 & * & 0 & 0 & & O_2 \\
* & 0 & 0 & 0 & & O_3 \\
0 & * & 1 & 1 & & O_4 \\
0 & * & 1 & 0 & & O_5 \\
0 & 0 & 0 & * & & O_6 \\
0 & * & 0 & 0 & & O_7 \quad .
\end{array}
$$

The proximities determined by this table are given by the following table:

i	indices of points proximate to O_i
0	$1, 2, 3$
1	$2, 3$
2	$3, 4, 5, 6, 7$
3	4
4	$5, 6$
5	$6, 7$
6	7

.

Corollary. *In addition to conditions $i) - iii)$ of Lemma 1.1, the proximity relation on any k-rational point sequence satisfies: $iv)$ for each j, there are at most d points O_i such that $O_j \to O_i$. Moreover, $i) - iv)$ are the only restrictions that apply. I.e., given a set of symbols $\{O^*_i \mid i = 0, 1, 2, \ldots\}$ and a relation $\stackrel{*}{\to}$ consistent with conditions $i) - iv)$, there is a k-rational point sequence $\{O_i \mid i = 0, 1, 2, \ldots\}$ such that $O_j \to O_i$ if and only if $O^*_j \stackrel{*}{\to} O^*_i$.*

Proof. Condition $iv)$ follows from the fact that there are only d choices for j in $\{a^{(i)}_j\}$ and that for each i an entry of the form $a^{(i)}_{j(i)} = 1 - x^{(i-1)}_{j(i)}$ must occur. As to the second statement, given a relation satisfying $(i - iv)$, it is easy to construct a coordinate table realizing that relation. /////

3. Virtual transforms and strict transforms.

Let $\{O_i\}$, $i=0,\ldots,n$, be a k-rational point sequence, $\{a_j^{(i)}\}$ a coordinate table for $\{O_i\}$. Let $x_1^{(i)},\ldots,x_d^{(i)}$ be the coordinates about O_i determined by the table as in the previous section and let $\mathcal{A}^{(i)} := k[x_1^{(i)},\ldots,x_d^{(i)}]$.

Here is some new notation that we will use frequently. For each i, let $S^{(i)}$ be a copy of $\mathbf{Z}^d$ and let $S_+^{(i)}$ be the submonoid $\mathbf{N}^d$ of $S^{(i)}$. If $\alpha \in S^{(i)}$, then $|\alpha| := \alpha_1 + \cdots \alpha_d$. If $f = \sum c_\alpha {x^{(i)}}^\alpha \in \mathcal{A}^{(i)}$, then $\mathrm{supp}^{(i)}(f) := \{\, \alpha \in S^{(i)} \mid c_\alpha \neq 0 \,\}$. This set is called the support of f. The order of f at O_i, denoted $\mathrm{ord}(i,f)$, is $\min\{\, |\alpha| \mid \alpha \in \mathrm{supp}^{(i)} \,\}$.

Let $\mathbf{m}^{(i)}$ denote the ideal in $\mathcal{A}^{(i)}$ consisting of all polynomials that vanish at O_i. Thus $\mathbf{m}^{(i)}$ consists of all polynomials in the variables $x_1^{(i)},\ldots,x_d^{(i)}$ with zero constant term, and $(\mathbf{m}^{(i)})^\ell$ consists of all polynomials with order at least ℓ. Note that $\mathrm{ord}(i,f) := \max\{\, \ell \mid f \in (\mathbf{m}^{(i)})^\ell \,\}$ and that that

$$\mathrm{ord}(i,f) \geq m \quad \Longleftrightarrow \quad \exists g \in \mathcal{A}^{(i+1)} \text{ such that } f = \varepsilon_i^m g. \tag{3.1}$$

Thus for any $f \in \mathcal{A}^{(i)}$, there is $g \in \mathcal{A}^{(i+1)} \setminus \varepsilon_i \mathcal{A}^{(i+1)}$ (*i.e.*, g is not divisible by ε_i) such that

$$f = \varepsilon_i^{\mathrm{ord}(i,f)} g. \tag{3.2}$$

Definition. *Suppose $f \in \mathcal{A}^{(s)}$. Given a sequence of integers $\nu = \{\nu_s, \nu_{s+1}, \ldots\}$, we define the virtual transforms of f, $V_s^{(i)}(\nu,f)$, for $i \geq s$ as follows:*

$$\begin{aligned} V_s^{(s)}(\nu,f) &:= f\,, \\ V_s^{(i+1)}(\nu,f) &:= \varepsilon_i^{-\nu_i} V_s^{(i)}(\nu,f) \\ &= \varepsilon_i^{-\nu_i}\varepsilon_{i-1}^{-\nu_{i-1}}\cdots\varepsilon_s^{-\nu_s} f. \end{aligned} \tag{3.3}$$

$V_s^{(i)}(\nu,f)$ may fail to belong to $\mathcal{A}^{(i)}$, but obviously $V_s^{(i)}(\nu,f)$ is always in the fraction field of $\mathcal{A}^{(i)}$. If $V_s^{(i)}(\nu,f) \in \mathcal{A}^{(i)}$ and the condition

$$\nu_i \leq \mathrm{ord}(i, V_s^{(i)}(\nu,f))$$

is satisfied, then $V_s^{(i+1)}(\nu,f)$ is in $\mathcal{A}^{(i+1)}$, by 3.2. If this condition is satisfied for $i = s,\ldots,n$, we say that f *satisfies the infinitely near base condition* $\{O_i,\nu_i\}_{i=s}^n$. In the next section, we examine the set of all f satisfying such a base condition. For remarks on the origins of this concept and its relations to other kinds of base conditions, see [Z], page 30.

Definition. *For $f \in \mathcal{A}^{(s)}$, the strict transforms of f are defined for $i \geq s$ by:*

$$\begin{aligned} T_s^{(s)}(f) &:= f, \\ T_s^{(i+1)}(f) &:= \varepsilon_i^{-\mu_{s,i}(f)} T_s^{(i)}(f), \end{aligned} \tag{3.4}$$

where

$$\mu_{s,i}(f) := \mathrm{ord}(i, T_s^{(i)}(f)). \tag{3.5}$$

Letting $\mu(f)$ denote the sequence $\{\mu_{s,s}(f), \mu_{s,s+1}(f), \ldots\}$, we have

$$T_s^{(i)}(f) = V_s^{(i)}(\mu(f), f).$$

Obviously, if $f \in \mathcal{A}^{(s)}$, then f satisfies the base condition $\{O_i, \mu_{s,i}(f)\}_{i=s}^n$ for any sequence $\{O_i\}_{i=s}^n$. We say that f *passes through* $O_s, \ldots, O_n$ if $T_s^{(i)}(f) \in \mathbf{m}^{(i)}$ for $i = s, \ldots, n$. This is equivalent to the condition

$$1 \leq \mu_{s,i}(f) \quad \text{for} \quad i = s, \ldots, n.$$

Lemma 3.2. *For any* $f \in \mathcal{A}^{(s)}$ *and any* $i \geq s$,

$$\mu_{s,i}(f) \geq \mu_{s,i+1}(f) \geq 0.$$

Proof. It is sufficient to prove this assuming that $s = i = 0$ and $j(1) = 1$, for all other cases reduce to this by changing indices. For convenience, write x in place of $x^{(0)}$, x' in place of $x^{(1)}$, T in place of $T_0^{(1)}$ and assume $\mu = \mathrm{ord}(0, f)$. Pick $\beta \in \mathrm{supp}^{(0)}(f)$ such that $(\beta_2, \ldots, \beta_d)$ is a maximal element (in the component-wise order) of

$$\{ (\alpha_2, \ldots, \alpha_d) \mid \exists\, \alpha_1 \ (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathrm{supp}^{(0)}(f) \ \& \ \alpha_1 + \ldots + \alpha_d \ = \mu \}.$$

Then $c_\beta (x'_2 + a_2)^{\beta_2} \cdots (x'_d + a_d)^{\beta_d}$ is a summand of $T(f)$, and no other summands involve the monomial ${x'_2}^{\beta_2} \cdots {x'_d}^{\beta_d}$. Thus, $\mathrm{ord}(1, T(f)) \leq \beta_2 + \cdots + \beta_d \leq \mu$. /////

4. Multiplicities and proximity.

In this section, we show that the relationship between the numbers $\mathrm{ord}(n, f)$ and $\mu_{s,n}(f)$ are completely determined by the proximity relations. Suppose a sequence $\{O_i\}_{i=0}^N$ has been fixed. For $s, i, n \in \{0, \ldots, N\}$, set

$$P(s,i) := \begin{cases} \mu_{s+1,i}(\varepsilon_s) & \text{if } s < i \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\omega(i,n) := \begin{cases} \mathrm{ord}(n, \varepsilon_i) & \text{if } i \le n \\ 0 & \text{otherwise.} \end{cases}$$

For reference, recall that (3.5) defines

$$\mu_{s,i}(f) := \mathrm{ord}(i, T_s^{(i)}(f)).$$

Proposition 4.1. *Suppose* $0 \le s < n$ *and* $f \in \mathcal{A}^{(s)}$. *Then*

$$P(s,n) = \begin{cases} 1, & \textit{if } O_n \to O_s; \\ 0, & \textit{otherwise,} \end{cases} \tag{i}$$

$$\mathrm{ord}(n, f) = \sum_{i=s}^{n} \mu_{s,i}(f)\omega(i,n), \tag{ii}$$

$$\omega(s,n) = \sum_{i=s+1}^{n} P(s,i)\omega(i,n). \tag{iii}$$

Proof. As in the proof of Lemma 2.1, to prove (i) it suffices to treat the case $s = 0$. Since $P(0,1) = \mu_{1,1}(\varepsilon_0) = 1$, the case $n = 1$ is clear. Suppose $n \ge 2$. If $O_n \to O_0$, then by Lemma 2.1

$$T_1^{(i)}(\varepsilon_0) = x_{j(1)}^{(i)} \quad \text{for } i = 2, \ldots, n.$$

From this it is immediate that $P(0,n) = 1$. If on the other hand $O_n \not\to O_0$, then choose i_0, $2 \le i_0 \le n$, to be the least index such that $O_{i_0} \not\to O_0$. Then either $T_1^{(i_0)}(\varepsilon_0) = 1$ (if $j(i_0) = j(1)$) or $T_1^{(i_0)}(\varepsilon_0) = x_{j(1)}^{(i_0)} + a_{j(1)}^{(i_0)}$, with $a_{j(1)}^{(i_0)} \ne 0$. In either case, $P(0, i_0) = 0$, so $P(0,n) = 0$ by Lemma 3.2.

Now we prove (ii). From 3.4, we obtain, for $f \in \mathcal{A}^{(s)}$,

$$f = T_s^{(n)}(f) \prod_{i=s}^{n-1} \varepsilon_i^{\mu_{s,i}(f)}.$$

By 2.2 and 2.3,

$$\omega(n,n) = 1,$$

and thus

$$\mathrm{ord}(n, f) = \sum_{i=s}^{n} \mu_{s,i}(f)\mathrm{ord}(n, \varepsilon_i), \tag{$*$}$$

which is what was to be proved.

To prove (iii), apply $(*)$ to $f = \varepsilon_{t-1} \in \mathcal{A}^{(t)}$, and then set $t = s + 1$. /////

Corollary 4.2. *Regard ω and P as square matrices of dimension $N+1$, and let I denote the identity matrix. Then*

$$\omega = (I - P)^{-1}.$$

Proof. One checks easily that 4.1.*iii* implies $\omega = P\omega + I$. /////

Remark. Corollary 4.2 originally appears in work of Du Val. For more discussion of this result, see [L-J] and [L$_2$]. Computationally, it is trivial to create a proximity matrix from a coordinate table. (See the collection of *Mathematica* programs that accompany these notes at my web site.)

Definition. *Suppose a sequence of points $O = \{O_i\}$ and a sequence of integers $\nu = \{\nu_i\}$ is given. Let*

$$I(O, \nu, s, n) := \{\, f \in \mathcal{A}^{(s)} \mid \nu_j \le \operatorname{ord}(j, V_s^{(j)}(\nu, f)) \quad \text{for} \quad s \le j \le n \,\}.$$

If $f \in \mathcal{A}^{(s)}$ and $f \in I(O, \nu, s, n)$, we say that f satisfies the base condition $\{O_j, \nu_j\}_{j=s}^{n}$. When $O = \{O_i\}_{i=s}^{n}$, we write $I(O, \nu)$ as an abbreviation for $I(O, \nu, s, n)$.

Proposition 4.3. *$I(O, \nu, s, n)$ is a complete ideal in $\mathcal{A}^{(s)}$.*

Proof. From the definition of the virtual transform, we get that for any $f \in \mathcal{A}^{(s)}$ and any $j = s, \dots n$:

$$\operatorname{ord}(j, V_s^{(j)}(\nu, f)) = \operatorname{ord}(j, f) - \sum_{i=s}^{j-1} \nu_i \omega(i, j).$$

Thus, for $j = s, \dots n$:

$$\nu_j \le \operatorname{ord}(j, V_s^{(j)}(\nu, f)) \Longleftrightarrow \sum_{i=s}^{j} \nu_i \omega(i, j) \le \operatorname{ord}(j, f).$$

This shows that $I(O, \nu, s, n)$ is an intersection of ord_i-ideals./////

Remarks. Recall that $\operatorname{ord}(j, f) = \sum_{i=0}^{j} \mu_i(f)\omega(i, j)$. Thus

$$\begin{aligned} \textstyle\sum_{i=0}^{j} \nu_i \omega(i, j) \le \operatorname{ord}(j, f) \text{ for } j = 0, \cdots, n \quad &\Longleftrightarrow \quad (\mu(f) - \nu)\omega \in \mathbf{N}^{n+1} \\ &\Longleftrightarrow \quad \mu(f) - \nu \in (\mathbf{N}^{n+1})(I - P). \end{aligned}$$

(Here, $\omega = (I - P)^{-1}$ is interpreted as an $n+1 \times n+1$ integer matrix.)

In the two-dimensional case, work of Enriques and Zariski shows that $I(O, \nu)$ is an ord_n-ideal if $\nu_0, \dots, \nu_n$ is a positive sequence that is minimal (in the term-wise ordering) among those satisfying the so-called "proximity inequalities" (see below). In this case, $I(O, \nu)$ is the simple complete ideal in $\mathcal{A}^{(0)}$ corresponding to the infinitely near point O_n, see [L-J]. In higher dimensions, I do not know the precise conditions under which $I(O, \nu)$

is an ord_n-ideal. (In any dimension, one may easily find examples where $I(O,\nu)$ is *not* an ord_n-ideal.)

Also note that the ideal $I(O,\nu)$ may fail to be finitely supported. In other words, even though $I(O,\nu)$ is determined by assigning data to finitely many points, the support of this ideal—see [L]—may not be finite. A major question is to understand how the support of $I(O,\nu)$ is related to the data $\{O_i,\nu_i\}$ defining it.

The chief goal of the present work is the analysis of $I = I(O,\nu)$. In particular, we want to be able to compute $\dim_k(I+\mathbf{m}^s)/\mathbf{m}^s$ for any O, ν and s.

5. Remarks on the 2-dimensional case.

The theory of infinitely near base conditions on curves on a smooth surface was described fully by Enriques and Chisini in [EC]. In 1938, as a testing ground for his valuation-theoretic formulation of algebraic geometry, Zariski reformulated this theory as a theory about valuation ideals in $k[x,y]$, [Z_2]. Later, Zariski presented a generalization that applies to arbitaray complete ideas in two-dimensional regular local rings, [ZS]. Among the theorems that Zariski proved in this context, one of the most influential was the unique factorization theorem for complete ideals. In the 1980s, there were attempts to extend unique factorization to higher dimensions, but counterexamples to a straightforward generalization were found.

Lejeune-Jalabert [L-J] and Lipman [L_2] have pointed out that the original formualtion of Enriques, which rested on the so-called "proximity inequalities", remains a useful way of looking at the theory. This actually applies to infinitely near base conditions in the plane corresponding to trees of points rather than just sequences. We provide a summary of the special case of Enriques' theory that applies to the kind of point sequences that we have been considering.

Let $\{O_i\}$ be a point sequence in the plane and let $\{\nu_i\}$ be a sequence of non-negative integers. We say that $\{\nu_i\}$ satisfies the *proximity inequalities* corresponding to $\{O_i\}$ if:

$$\text{for each } i,\, \nu_i \geq \sum_{O_j \to O_i} \nu_j .$$

Theorem. *(See [L-J], p. 354.) For any $f \in \mathcal{A}^{(0)}$, if f passes through the $\{O_i\}$, then the multiplicities $\mu_i := \mu_{0,i}(f)$ satisfy the proximity inequalities corresponding to $\{O_i\}$. Conversely, if $\{\nu_i\}$ satisfies the proximity inequalities corresponding to $\{O_i\}$, then there is $f \in \mathcal{A}^{(0)}$ such that $\mu_{0,i}(f) = \nu_i$.*

The hard part of this theorem is the second statement. Later in these notes, we will discuss this further. For the time being, we will look at some concrete implications of the proximity inequalities. First, note that having assigned multiplicity 1 to the last point in a sequence, the minimal multiplicities of preceeding points are determined inductively by the relation $\nu_i = \sum_{O_j \to O_i} \nu_j$. The coordinate table provides a convenient display. In each column below, we assigned a multiplicity of 1 to a point and 0 to all succeeding points. The minimal multiplicities compatible with the proximity inequalities are shown:

										O_i
$-$	$-$	10	6	4	2	2	2	1	1	O_0
0	$*$	5	3	2	1	1	1	1	0	O_1
$*$	0	5	3	2	1	1	1	0	0	O_2
$*$	1	5	3	2	1	1	0	0	0	O_3
$*$	0	2	1	1	1	0	0	0	0	O_4
0	$*$	2	1	1	0	0	0	0	0	O_5
0	$*$	1	1	0	0	0	0	0	0	O_6
$*$	0	1	0	0	0	0	0	0	0	O_7 .

It is easy to see that any multiplicity sequence that is compatible with the proximity inequalities is a unique sum of the columns shown. This being true for any table, it follows that:

Lemma. *The set of all non-negative multiplicity sequences on* $\{ O_i \mid 0 \leq i \leq n \}$ *satisfying the proximity inequalities forms a commutative monoid isomorphic to* $\mathbf{N}^{n+1}$.

Zariski [Z_2] considered the monoid of ord_n valuation ideals, where the operation is ideal multiplication. This monoid is isomorphic to the one we have just described. For a discussion of monoids of ideas related to sequences infinitely near points in any dimension, see [L_1]. Additional discussion of monoids is in [L-J].

6. Exponent space and monomial transforms

The monomials form a richly structured basis for the k-vector space, $k[x_1, \ldots, x_d]$. Computational methods in commutative algebra that use Gröbner bases or sparse resultants make effective use of this structure. Also, toric geometry is largely dependent upon useful structures that can be defined in the space of monomials. My goal is to develop ways to use monomial structures to address the problem of finding polynomials that exactly satisfy given base conditions. In this quest, the first problem seems to be to develop effective nomenclature. This section begins an exploration in that direction.

6.1. Order structure in $\mathbf{Z}^d$. Suppose E is a copy of $\mathbf{Z}^d$. Let $\pi_i : E \to \mathbf{Z}$ be the projection onto the i^{th} component. Let $\leq$ denote the componentwise order on E—that is, $u \leq v$ iff $\pi_i(u) \leq \pi_i(v)$ (in the usual order on $\mathbf{Z}$) for each $i = 1, \ldots, d$. Let $E_+ := \{ u \in E \mid 0 \leq u \}$. Then

$$u \leq x \;\Leftrightarrow\; x - u \in E_+ \;\Leftrightarrow\; x \in u + E_+.$$

The following lemma summarizes the most important facts about lower bounds of subsets of E. The proof is easy, so we do not include it.

6.1.1. Lemma. *Suppose* $X \subseteq E$ *and* $u \in E$. *Let* $\pi_i(X) := \{ \pi_i(x) \mid x \in X \}$.

a) u *is a lower bound for* $X \Leftrightarrow X \subseteq u + E_+ \Leftrightarrow X - u \subseteq E_+$.
b) X *has a lower bound in* E *if and only if each* $\pi_i(X)$ *has a lower bound in* $\mathbf{Z}$.
c) If X *has a lower bound in* E *then* X *has a greatest lower bound in* E.

d) The following are equivalent:

i) γ *is the greatest lower bound of* X*;*

ii) $\pi_i(\gamma) = \min\{\, \pi_i(x) \mid x \in X \,\}$*;*

iii) $X - \gamma \subseteq E_+$ *and* $X - \gamma$ *includes points on each face of* E_+*;*

iv) the set of all lower bounds of X *is* $\gamma - E_+$*.*

e) Denoting the greatest lower bound of X *by* $\wedge X$*, we have* $\wedge(u+X) = u + \wedge X$*.*/////

6.2. Isomorphisms and strengthening the order. Let E' be another copy of $\mathbf{Z}^d$ and let $\phi : E \to E'$ be a group isomorphism such that $\phi(E_+)$ is properly contained in ${E'}_+$ (equivalently, the matrix representing ϕ with respect to the standard bases in E and E' has non-negative entries). Such a ϕ satisfies:

$$\text{for all } u, v \in E,\ u \le v \ \Rightarrow\ \phi(u) \le \phi(v).$$

The order on E', however, is properly stronger than the order induced from E by ϕ, so the converse is not generally true.

Suppose that $B \subseteq E$ and set $B' := \phi(B)$. Assume B has a greatest lower bound $\gamma \in E$. Then $\phi(\gamma)$ is a lower bound of B' in E', so B' has a greatest lower bound in E', call it γ'. Clearly, $\phi(\gamma) \le \gamma'$ in E'. When $d = 2$, it is also the case that $\gamma \le \phi^{-1}(\gamma')$ in E, as can be seen by an elementary diagram. However, when $d > 2$ this may fail.

6.2.1. Example. Let $d = 3$ and suppose $\phi(u) = u \cdot A$, where

$$A = \begin{Bmatrix} 2 & 0 & 1 \\ 4 & 1 & 1 \\ 3 & 2 & 0 \end{Bmatrix} \quad \text{and} \quad A^{-1} = \begin{Bmatrix} -2 & 2 & -1 \\ 3 & -3 & 2 \\ 5 & -4 & 2 \end{Bmatrix}.$$

Also, let $B = \{\, (1,0,0), (0,1,0), (0,0,1) \,\}$. Then $\gamma = (0,0,0)$ in E and $\phi(\gamma) \le \gamma' = (2,0,0) \in E'$. But $\phi^{-1}(\gamma') = (-4, 2, -1)$, which is not comparable to γ in E. /////

Suppose $d = 2$ and $B' - \gamma' = \{\, (0,q), (p,0) \,\} \subseteq E'$. Suppose $\phi(u) := u \cdot A$, where $A = \begin{Bmatrix} a & b \\ c & d \end{Bmatrix}$. We are assuming, of course, that $a, b, c, d \ge 0$ and $\det(A) = 1$. Now,

$$\phi^{-1}(B' - \gamma') = (B' - \gamma') \cdot A^{-1} = \{\, q(-c, a), p(d, -b) \,\}.$$

Let $v := \phi^{-1}(\gamma') - \gamma$. In other words, $-v$ is the greatest lower bound of $\phi^{-1}(B' - \gamma') = B - \phi^{-1}(\gamma')$ in E. From the above, we see $v = (qc, pb)$. Note that

$$\begin{aligned} B - \gamma = \phi^{-1}(B' - \gamma') + v &= \{\, q(-c, a), p(d, -b) \,\} + (qc, pb) \\ &= \{\, (0, qa + pb), (qc + pd, 0) \,\}. \end{aligned}$$

Observing that

$$A \cdot \begin{Bmatrix} q \\ p \end{Bmatrix} = \begin{Bmatrix} qa + pb \\ qc + pd \end{Bmatrix},$$

we have a formula that enables us to pass from data determining $B' - \gamma'$ to data determining $B - \gamma$.

6.2.2. Example. Let $M_{0*} := \begin{Bmatrix} 1 & 1 \\ 0 & 1 \end{Bmatrix}$ and $M_{*0} := \begin{Bmatrix} 1 & 0 \\ 1 & 1 \end{Bmatrix}$. If ϕ is determined by M_{0*}^k, then $v = (0, kp)$ and $B - \gamma = \{\, (0, kp+q), (p, 0) \,\}$. If ϕ is determined by M_{*0}^k, then $v = (kq, 0)$ and $B - \gamma = \{\, (0, q), (p + kq, 0) \,\}$. /////

6.3. Sequences. We shall extend the discussion to a sequence of isomorphisms. Suppose

$$E^{(0)} \xrightarrow{\phi^{(1)}} E^{(1)} \xrightarrow{\phi^{(2)}} \cdots \xrightarrow{\phi^{(n)}} E^{(n)}$$

has been given, each satisfying $\phi^{(i)}(E_+^{(i-1)}) \subseteq E_+^{(i)}$. Suppose $B = B^{(0)} \subseteq E^{(0)}$. Let $B^{(i)}$ be the image of B in $E^{(i)}$. Assume that $B^{(0)}$ has a greatest lower bound $\gamma_0 \in E^{(0)}$. Then the image of γ_0 in $E^{(i)}$, which we denote $\gamma_0^{(i)}$, is a lower bound for each $B^{(i)}$. Hence each each $B^{(i)}$ has a greatest lower bound in $E^{(i)}$; we denote this γ_i. Of course, $B^{(i)} - \gamma_i$ is in $E_+^{(i)}$ and meets each face of $E_+^{(i)}$. We denote the image of γ_i in $E^{(j)}$ by $\gamma_i^{(j)}$. For each $j = 1, \ldots, n$:

$$\gamma_0^{(j)} \le \gamma_1^{(j)} \le \cdots \le \gamma_{j-1}^{(j)} \le \gamma_j^{(j)} \text{ (in } E^{(j)}\text{)}.$$

6.3.1. Definition. *For $i = 0, \ldots, n-1$ let $v_i := \gamma_{i+1}^{(i)} - \gamma_i$. (We do not define v_n.)*

Note that $v_i \in E^{(i)}$. As usual, we denote the image of v_i in $E^{(j)}$ by $v_i^{(j)}$. When $0 \le j < k \le n$, we have:

$$\begin{aligned} \gamma_k^{(i)} - \gamma_j^{(i)} &= (\gamma_k^{(i)} - \gamma_{k-1}^{(i)}) + (\gamma_{k-1}^{(i)} - \gamma_{k-2}^{(i)}) + \cdots + (\gamma_{j+1}^{(i)} - \gamma_j^{(i)}) \\ &= v_{k-1}^{(i)} + v_{k-2}^{(i)} + \cdots + v_j^{(i)} . \end{aligned}$$

In particular:

$$\gamma_n^{(i)} - \gamma_j^{(i)} = v_{n-1}^{(i)} + \cdots + v_j^{(i)} .$$

6.3.2. Example. We treat the case when $d = 2$, $B^{(n)} - \gamma_n = \{\, (0, q), (p, 0) \,\}$ and each $\phi^{(i)}$ is given by multiplying on the right by either M_{0*} or M_{*0}. (These matrices were defined in the previous example.) The entire sequence, then, can be written as:

$$E^{(0)} \xrightarrow{M_1} E^{(n_1)} \xrightarrow{M_2} E^{(n_2)} \xrightarrow{M_3} \cdots \xrightarrow{M_s} E^{(n_s)} = E^{(n)},$$

where each $M_j = M^{k_j}$, with M being one of the matrices M_{*0} or M_{0*} and $k_j := n_j - n_{j-1}$. (We assume that successive M_j are not powers of the same matrix.) For definiteness, let us suppose that M_s is a power of M_{0*}. The following table exhibits the $B^{(i)} - \gamma_i$ and the

v_i working backwards from $E^{(n)}$. We write $B^{(i)} - \gamma_i$ as $\{ (0, q_i), (p_i, 0) \}$ and show q_i and p_i only.

i	q_i	p_i	v_i
n_s	q	p	$***$
$n_s - 1$	$p+q$	p	$(0,p)$
$n_s - 2$	$2p+q$	p	$(0,p)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n_{s-1}	$k_s p + q$	p	$(0,p)$
$n_{s-1} - 1$	$k_s p + q$	$(k_s p + q) + p$	$(k_s p + q, 0)$
$n_{s-1} - 2$	$k_s p + q$	$2(k_s p + q) + p$	$(k_s p + q, 0)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n_{s-2}	$k_s p + q$	$k_{s-1}(k_s p + q) + p$	$(k_s p + q, 0)$
$n_{s-2} - 1$	$p_{n_{s-2}} + q_{n_{s-2}}$	$p_{n_{s-2}}$	$(0, p_{n_{s-2}})$

The pattern is apparent. In the interval $\{ i \mid n_{j-1} \le i \le n_j \}$, either q_i or p_i is constant, depending on whether M_{0*} or M_{*0} is being applied. If p_i is constant (*i.e.*, M_{0*} is being applied) then $q_i - q_{i-1} = p_i$, when $n_{j-1} < i \le n_j$.

These patterns are closely related to the proximity inequalities. Indeed, let us write an *arbitrary* coordinate table as below. We do not demand that all the transforms be monomial. Mark the last row with the pair (p, q), and if a row is marked (i, j), then mark the preceeding row according to the following rules:

$$\begin{array}{cccc} - & - & i+j, & j \\ * & 0 & i & j \end{array} \qquad \begin{array}{cccc} - & - & j & j \\ * & 1 & i & j \end{array}$$

$$\begin{array}{cccc} - & - & i & i+j \\ 0 & * & i & j \end{array} \qquad \begin{array}{cccc} - & - & i & i \\ 1 & * & i & j \end{array}$$

Note that the last rule will not be used in tables such as we construct, but it may apply in more general tables. The rules for the column labelled ν are stated after the table.

		P	Q	ν
$-$	$-$	$14p+7q$	$8p+4q$	$8p+4q$
$*$	0	$6p+3q$	$8p+4q$	$6p+3q$
0	$*$	$6p+3q$	$2p+q$	$2p+q$
$*$	0	$4p+2q$	$2p+q$	$2p+q$
$*$	0	$2p+q$	$2p+q$	$2p+q$
$*$	1	$7p+3q$	$2p+q$	$2p+q$
$*$	0	$5p+2q$	$2p+q$	$2p+q$
$*$	0	$3p+q$	$2p+q$	$2p+q$
$*$	0	p	$2p+q$	p
0	$*$	p	$p+q$	p
0	$*$	p	q	q .

The last entry in the ν column is the letter in the position of the asterisk. Each other entry is the minimum of the two entries to the left.

Lemma. *In any table constructed according to the above rules, the entries in the ν column obey the proximity equalities.*

6.3.3. Example. We treat the case when $d = 3$ and each $\phi^{(i)}$ is given by multiplying on the right by one of the matrices M_{*00}, M_{0*0} or M_{00*}. We simply exhibit the data that accompanies an arbitrarily chosen sequence of matrices. One should think of the $B^{(i)} - \gamma_i$ as the support of a trinomial. In the table below, $B^{(0)} - \gamma_0$ represents the trinomial $ax^{12}y^5 + by^8z^4 + cx^5z^8$. Note that the non-zero entry in v_i is the order of $B^{(0)} - \gamma_0$. The successive $B^{(i)} - \gamma_i$ are the proper transforms of $B^{(0)} - \gamma_0$ along the sequence of infinitely near points associated with the table in the leftmost column. The v_i are the minimal multiplicities with which a generic polynomial that ultimately transforms to a generic polynomial with support $B^{(n)} - \gamma_n$ vanishes along the sequence.

i	$M^{(i)}$	$B^{(i)}$	$B^{(i)} - \gamma_i$	γ_i	v_i
0	$-\ -\ -$	$\begin{matrix} 4 & 1 & -3 \\ -8 & 4 & 1 \\ -3 & -4 & 5 \end{matrix}$	$\begin{matrix} 12 & 5 & 0 \\ 0 & 8 & 4 \\ 5 & 0 & 8 \end{matrix}$	$-8\ \ -4\ \ -3$	$12\ \ 0\ \ 0$
1	$*\ 0\ 0$	$\begin{matrix} 2 & 1 & -3 \\ -3 & 4 & 1 \\ -2 & -4 & 5 \end{matrix}$	$\begin{matrix} 5 & 5 & 0 \\ 0 & 8 & 4 \\ 1 & 0 & 8 \end{matrix}$	$-3\ \ -4\ \ -3$	$0\ \ 9\ \ 0$
2	$0\ *\ 0$	$\begin{matrix} 2 & 0 & -3 \\ -3 & 2 & 1 \\ -2 & -1 & 5 \end{matrix}$	$\begin{matrix} 5 & 1 & 0 \\ 0 & 3 & 4 \\ 1 & 0 & 8 \end{matrix}$	$-3\ \ -1\ \ -3$	$0\ \ 0\ \ 6$
3	$0\ 0\ *$	$\begin{matrix} 2 & 0 & -1 \\ -3 & 2 & 0 \\ -2 & -1 & 2 \end{matrix}$	$\begin{matrix} 5 & 1 & 0 \\ 0 & 3 & 1 \\ 1 & 0 & 3 \end{matrix}$	$-3\ \ -1\ \ -1$	$4\ \ 0\ \ 0$
4	$*\ 0\ 0$	$\begin{matrix} 1 & 0 & -1 \\ -1 & 2 & 0 \\ -1 & -1 & 2 \end{matrix}$	$\begin{matrix} 2 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{matrix}$	$-1\ \ -1\ \ -1$	$0\ \ 3\ \ 0$
5	$0\ *\ 0$	$\begin{matrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 2 \end{matrix}$	$\begin{matrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{matrix}$	$-1\ \ 0\ \ -1$	$0\ \ 0\ \ 2$
6	$0\ 0\ *$	$\begin{matrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{matrix}$	$\begin{matrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$	$-1\ \ 0\ \ 0$	$1\ \ 0\ \ 0$
7	$*\ 0\ 0$	$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$	$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$	$0\ \ 0\ \ 0$	$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$

References

[EC] F. Enriques and O. Chisini, *Lezzioni sulla teoria geometrica delle equazioni e delle funzioni algebriche, vol. II*, N. Zanichelli, Bologna, 1918.

[L-J] M. Lejeune-Jalabert, *Linear systems with infinitely near base conditions and complete ideals in dimension two*, Singularity Theory, 1991, (D. T. Lê, K. Saito & B. Teissier, editors), World Scientific, Singapore, 1995, 345–369.

[L_1] J. Lipman, *On complete ideas in regular local rings*, Algebraic Geometry and Commutative Algebra, vol. I, in honor of Masayoshi Nagata, Kinokuniya, Tokyo, 1988, pp. 203–231.

[L_2] J. Lipman, *Proximity inequalities for complete ideals in two-dimensional regular local rings*, Commutative Algebra: Syzygies, Multiplicities and Birational Algebra, 1992, (W. Heinzer, C. Huneke and J. Sally, editors), Contemporary Mathematics **159**, American Mathematical Society, Providence, 1994, 293–306.

[Z] O. Zariski, *Algebraic Surfaces*, Springer-Verlag 1971.

[Z_2] O. Zariski, Polynomial ideals defined by infinitely near base points, American J. Math. **60** (1938), 151–204.

[ZS] O. Zariski and P. Samuel, *Commutative Algebra, vol. 2*, D. van Nostrad, Princeton, 1960.