Mathematical Foundations for the Common Core
A Course for Middle and Secondary Teachers

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Topic: Expressions and Equations

Section A. François Viète and the basic ideas of algebra

François Viète was born in 1540 in Poitou in west-central France. Viète was educated as a lawyer, and in the 1570s, having established a good reputation, he was invited by King Henry III to act as an adviser to the court in Paris. He served both Henry III and his successor Henry IV as privy councillor until his retirement in 1602. He died in 1603. The region of Viète's birth is also the region from which the Acadians emigrated to eastern Canada, beginning in 1604. The Cajuns of Louisiana are refugees of the British persecution of the Acadians in 1755–1763.

Viète studied and wrote about mathematics as a hobby. He is credited as the first mathematician to use letters as variables. One of his works, the *Five Books of Zetetica* (which was first published in 1591), contains numerous problems and solutions that illustrate his methods. All of the problems that Viète examines in this work are vivid examples of the some of the kinds of algebraic reasoning that the *Common Core State Standards for Mathematics* encourages.

We find in the *Zetetica*, numerous "word problems" in which one is asked to exhibit procedures for finding unknown quantities from information about known quantities derived from the unknowns. For example, the very first problem asks how to find two numbers, if their sum and difference is given. If the first level of generality in algebra is the use of letters in place of numbers to state and solve equations (e.g., find x and y if $x + y = 31$ and $x - y = 7$), then the *Zetetica* move to the next level, where classes of equations with parameters are considered (e.g., find x and y in terms of S and D, if $x + y = S$ and $x - y = D$). This forces greater attention to the *structure* of the problem at hand. In every case, the key to the solution lies in identifying

- the quantities involved,
- the operations performed upon them, and
- the relationships among the resulting quantities.

Solutions are found by

- choosing letters to represent the quantities,
- representing the operations using expressions, and
- translating the relationships into equations.

The equations once obtained are solved. This may be routine, or may require some imagination and originality.

Before looking at the problems themselves, let me draw your attention to the following points from overview of high school algebra in the *Common Core Standards*:

- *An expression is a record of a computation . . .. Creating an expression that describes a computation involving a general quantity requires the ability to express the computation in general terms, abstracting from specific instances.*

- *An equation is a statement of equality between two expressions, often viewed as a question asking for which values of the variables the expressions on either side are in fact equal.*

The patterns of reasoning required to work the *Zetetica* are reflected in the Agebra domain (A-CED) found on page 65. This domain bears a star, indicating that it is also a standard for the modeling strand. Here, I quote it in abridged form:

Create equations that describe numbers or relationships[*]

1. *Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear and quadratic functions, . . .*

2. *Create equations in two or more variables to represent relationships between quantities. . .*

3. *Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or nonviable options in a modeling context. . . .*

4. *Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. . . .*

Section B. Problems

Zetetica: Book I (Linear relations)

1.1 Given the sum and the difference of two numbers, find the numbers.

Solution. Let the sum be denoted S and let the difference be denoted D. We are asked to find two numbers x and y such that $S = x + y$ and $D = x - y$. If we add the two equations together, we find $S + D = 2x$, so $x = (S + D)/2$. If we subtract, we find $S - D = 2y$, so $y = (S - D)/2$.

1.2 Given the difference between two numbers and their ratio, find the numbers.

1.3 Given the sum of two numbers and their ratio, find the numbers.

1.4 Given two numbers that are less than a third number and the ratio of their deficiencies, find the third number.

1.5 Given two numbers that are greater than a third number and the ratio of their surpluses, find the third number.

1.6 Given two numbers, one greater than a third number and one less, and the ratio of the deficiency to the surplus, find the third number.

1.7 Divide a given length in such a way that a predefined part of the first segment plus a predefined part of the second will equal a prescribed number.

1.8 Divide a given length in such a way that a predefined part of the first segment subtracted from a predefined part of the second will equal a prescribed number.

1.9 Find two numbers with prescribed difference such that a predefined part of the first segment plus a predefined part of the second will equal a prescribed number.

1.10 Find two numbers with prescribed difference such that a predefined part of the first segment minus a predefined part of the second will equal a prescribed number.

Zetetica: Book II (Quadratic and cubic relations)

2.1 Given the product and the ratio of two numbers, find the numbers.

2.2 Given the product of two numbers and the sum of their squares, find the numbers.

2.3 Given the product of two numbers and their difference, find the numbers.

2.4 Given the product of two numbers and their sum. find the numbers.

2.5 Given the difference between two numbers and the sum of their squares, find the numbers.

2.6 Given the sum of two numbers and the sum of their squares, find the numbers.

2.7 Given the difference between two numbers and the difference between their squares, find the numbers.

2.8 Given the sum of two numbers and the difference between their squares, find the numbers.

2.9 Given the product of two numbers and the difference between their squares, find the numbers.

2.10 Given i) the sum of the squares of two numbers plus their product (*i.e.*, $a^2 + b^2 + ab$) and ii) one of the numbers, find the other number.

2.11 Given i) the sum of the squares of two numbers plus their product and ii) the sum of the numbers, find the numbers.

2.12 Given i) the sum of the squares of two numbers plus their product (*i.e.*, $a^2 + b^2 + ab$) and ii) the product itself, find the other number.

2.13 Given the sum of the squares of two numbers and the difference of the squares, find the numbers.

2.14 Given the sum of the cubes of two numbers and the difference of the cubes, find the numbers.

2.15 Given the difference between the cubes of two numbers and the product of the numbers, find the numbers.

2.16 Given the sum of the cubes of two numbers and the product of the numbers, find the numbers.

2.17 Given the difference between the cubes of two numbers and the difference between the numbers, find the numbers. *(Four times the difference between the cubes minus the cube of the difference, if divided by three times the difference between the numbers, yields the square of the sum of the numbers.)*

2.18 Given the sum of the cubes of two numbers and the sum of the numbers, find the numbers. *(See 2.17.)*

2.19 Solve 17 another way. *(The difference between the cubes minus the cube of the difference, if divided by three times the difference between the numbers, yields the product of the numbers.)*

2.20 Solve 18, another way. *(See 2.19.)*

2.21 Find a and b in terms of $D = (a - b)(a^2 - b^2)$ and $S = (a + b)(a^2 + b^2)$.

2.22 Find a and b in terms of $S = a^2 + b^2$ and $R = \frac{a\,b}{(a-b)^2}$.