Mathematical Foundations for the Common Core
A Course for Middle and Secondary Teachers

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Topic: Measurement

Section A. Reading and Interpreting the K-5 Measurement Strand

The measurement strand of the *Common Core State Standards for Mathematics* includes a progression of learning goals that spans Kindergarten to Grade Five. By the end of Grade Five, students are expected to be able to measure length, area, volume, mass, time and angle, convert between different units and interpret the meaning of arithmetic operations applied to measure numbers.

In Kindergarten and Grade One, students are expected to grasp the following ideas, which are foundational for all that follows.

i) Objects have measurable attributes of various kinds, such as length or weight.

ii) Two objects may be compared to determine whether or not they are equal with respect to a measurable attribute that they share, and if they are unequal, to find which is greater.

iii) One object may be measured by another with respect to an attribute that they share in common.

iv) The act of measurement is a systematic comparison that requires assigning different roles to the two objects that are being compared: one object *gets measured* and the other *does the measuring*. The latter is called the *unit*.

v) Measuring involves counting the number of units that "fit within" the object being measured, where the meaning of "fit within" depends on the attribute being measured. *In the case of length, one counts how many units may be placed end-to-end along the object being measured without exceeding it. In the case of weight, one may place the object being measured in one pan of a balance, and then count how many replicas of the unit may be placed in the other pan before the balance tips.*

vi) An act of measuring produces a number that characterizes the relationship of the object measured to the unit. *The Common Core Standards refers to "numbers with units," (page 58), which arise from measurement, and are called "quantities." Phrases such as "*24 *miles" or "*45 *miles per hour" are meant. We will use the term "measure number" to refer to the number that a measurement produces. Clearly, one needs to know the* unit *and the* thing measured *in order for a measure number to be meaningful.*

In subsequent grades, the principles of measurement are extended to a variety of quantitative attributes, including time, area, volume, mass and angle. Other ideas that are built upon the foundational ideas of Kindergarten and First Grade include the use of fractional units, conversions between units, and the manner in which arithmetic operations applied to measure numbers reflect physical operations that are performed on objects that are measured. In Grade Two, students are expected to become familiar with and use standard units of length (such as inches, feet, centimeters, meters), and to understand and use some basic connections between measurement and arithmetic. In Grade 3, students extend the ideas of measurement to time, liquid measure (capacity) and mass. In Grade 4, connections between measurement and arithmetic are developed, with a focus on making inferences about measurable attributes by using arithmetic to convert from one unit to another or to pass from measures of length to measures of area. Angle measure is also introduced, with explicit attention to the analogies to length measure. In Grade 5, the themes from Grade 4 are developed further, brining the full extent of the arithmetic learned up to this point to bear on situations involving measurement. Students do not merely solve problems. They are expected to provide reasoned accounts of manner in which arithmetic operations *represent* physical operations, e.g., relating volume to length and area via multiplication and relating addition of measure numbers to the physical combination of measured objects.

After Grade Five, the core ideas in the measurement stand are developed further in the context of Ratios and Proportional Relationships, which are studied in Grades Six and Seven, and in the development of ideas about number systems in these grades and beyond. Measurement error is studied in in relation to statistics, but the connections between the this and the ideas in the K-5 measurement strand are more superficial than the really deep ideas that tie together measurement, ratio, proportion and the real number system.

Text from the standards

In the following, we quote the text of the *Common Core Standards* for the Measurement and Data Strand.

Kindergarten

1. *Describe measurable attributes of objects, such as length or weight. Describe several measurable attributes of a single object.*
2. *Directly compare two objects with a measurable attribute in common, to see which object has "more of"/"less of" the attribute, and describe the difference. For example, directly compare the heights of two children and describe one child as taller/shorter.*

Grade One

1. *Order three objects by length; compare the lengths of two objects indirectly by using a third object.*
2. *Express the length of an object as a whole number of length units, by laying multiple copies of a shorter object (the length unit) end to end; understand that the length measurement of an object is the number of same-size length units that span it with no gaps or overlaps. Limit to contexts where the object being measured is spanned by a whole number of length units with no gaps or overlaps.*

Grade Two

1. *Measure the length of an object by selecting and using appropriate tools such as rulers, yardsticks, meter sticks, and measuring tapes.*
2. *Measure the length of an object twice, using length units of different lengths for the two measurements; describe how the two measurements relate to the size of the unit chosen.*
3. *Estimate lengths using units of inches, feet, centimeters, and meters.*
4. *Measure to determine how much longer one object is than another, expressing the length difference in terms of a standard length unit.*
5. *Use addition and subtraction within* 100 *to solve word problems involving lengths that are given in the same units, e.g., by using drawings (such as drawings of rulers) and equations with a symbol for the unknown number to represent the problem.*
6. *Represent whole numbers as lengths from 0 on a number line diagram with equally spaced points corresponding to the numbers* $0, 1, 2, \ldots,$ *and represent whole-number sums and differences within* 100 *on a number line diagram.*

Grade 3

1. *Tell and write time to the nearest minute and measure time intervals in minutes. Solve word problems involving addition and subtraction of time intervals in minutes, e.g., by representing the problem on a number line diagram.*
2. *Measure and estimate liquid volumes and masses of objects using standard units of grams (g), kilograms (kg), and liters (l). Add, subtract, multiply, or divide to solve one-step word problems involving masses or volumes that are given in the same units, e.g., by using drawings (such as a beaker with a measurement scale) to represent the problem.*

Grade 4

1. *Know relative sizes of measurement units within one system of units including km, m, cm; kg, g; lb, oz.; l, ml; hr, min, sec. Within a single system of measurement, express measurements in a larger unit in terms of a smaller unit. Record measurement equivalents in a two-column table. For example, know that 1 ft is 12 times as long as 1 in. Express the length of a 4 ft snake as 48 in. Generate a conversion table for feet and inches listing the number pairs* $(1, 12)$, $(2, 24)$, $(3, 36)$, $\ldots$.
2. *Use the four operations to solve word problems involving distances, intervals of time, liquid volumes, masses of objects, and money, including problems involving simple fractions or decimals, and problems that require expressing measurements given in a larger unit in terms of a smaller unit. Represent measurement quantities using diagrams such as number line diagrams that feature a measurement scale.*
3. *Apply the area and perimeter formulas for rectangles in real world and mathematical problems. For example, find the width of a rectangular room given the area of the flooring and the length, by viewing the area formula as a multiplication equation with an unknown factor.*
4. *Make a line plot to display a data set of measurements in fractions of a unit (1/2, 1/4, 1/8). Solve problems involving addition and subtraction of fractions by using information presented in line plots. For example, from a line plot find and interpret the difference in length between the longest and shortest specimens in an insect collection.*
5. *Recognize angles as geometric shapes that are formed wherever two rays share a common endpoint, and understand concepts of angle measurement:*
   a. *An angle is measured with reference to a circle with its center at the common endpoint of the rays, by considering the fraction of the circular arc between the points where the two rays intersect the circle. An angle that turns through* $1/360$ *of a circle is called a "one-degree angle," and can be used to measure angles.*
   b. *An angle that turns through* n *one-degree angles is said to have an angle measure of* n *degrees.*
6. *Measure angles in whole-number degrees using a protractor. Sketch angles of specified measure.*
7. *Recognize angle measure as additive. When an angle is decomposed into non-overlapping parts, the angle measure of the whole is the sum of the angle measures of the parts. Solve addition and subtraction problems to find unknown angles on a diagram in real world and mathematical problems, e.g., by using an equation with a symbol for the unknown angle measure.*

Grade 5

1. *Convert among different-sized standard measurement units within a given measurement system (e.g., convert* 5 *cm to* 0.05 *m), and use these conversions in solving multi-step, real world problems.*

2. *Make a line plot to display a data set of measurements in fractions of a unit (1/2, 1/4, 1/8). Use operations on fractions for this grade to solve problems involving information presented in line plots. For example, given different measurements of liquid in identical beakers, find the amount of liquid each beaker would contain if the total amount in all the beakers were redistributed equally.*
3. *Recognize volume as an attribute of solid figures and understand concepts of volume measurement.*
   a. *A cube with side length 1 unit, called a "unit cube," is said to have "one cubic unit" of volume, and can be used to measure volume.*
   b. *A solid figure which can be packed without gaps or overlaps using n unit cubes is said to have a volume of n cubic units.*
4. *Measure volumes by counting unit cubes, using cubic cm, cubic in, cubic ft, and improvised units.*
5. *Relate volume to the operations of multiplication and addition and solve real world and mathematical problems involving volume.*
   a. *Find the volume of a right rectangular prism with whole-number side lengths by packing it with unit cubes, and show that the volume is the same as would be found by multiplying the edge lengths, equivalently by multiplying the height by the area of the base. Represent threefold whole-number products as volumes, e.g., to represent the associative property of multiplication.*
   b. *Apply the formulas $V = l \times w \times h$ and $V = b \times h$ for rectangular prisms to find volumes of right rectangular prisms with whole-number edge lengths in the context of solving real world and mathematical problems.*
   c. *Recognize volume as additive. Find volumes of solid figures composed of two non-overlapping right rectangular prisms by adding the volumes of the non-overlapping parts, applying this technique to solve real world problems.*

Problems

1. List all the different kinds of things that students learn to measure by fifth grade and the grades in which they study them.
2. When you measure the length of a desk with the span of your hand, in addition to the desk itself and your hand, various multiples of your hand length are created. Describe the procedure precisely. What comparisons do you make? How do you know when to stop? How does this lead to a number? (This problem is asking you to take extraordinary care in making the descriptions precise, as in the well-known "explain to a space alien how to make a peanut butter and jelly sandwich" exercise.)
3. "Capacity" refers to the amount of liquid that fits in a container. A tablespoon is a common measure of capacity. Using a tablespoon, how would you measure the capacity of a drinking glass? In what ways is measuring capacity similar to measuring length. Break this down into the precise steps that are followed in both cases, and make a systematic analogy.
4. When a measurement is made, one object *gets measured* and another object *does the measuring*. When we divide numbers, one number *gets divided* and another number *does the dividing*. Is there a significant analogy here, or is the similarity coincidental? Explain.

Section B. Core Mathematical Concepts

What the Common Core expects students to learn is not a collection of useful facts, but a tight, coherent structure that is held together by very deep principles with ancient historical roots. This section outlines some fundamental concepts that motivate and make sense of the CCSS Measurement Strand. The ideas here were first written about by the ancient Greeks, and these ideas have become the starting point for a tremendous amount of mathematical thinking leading up to our modern understanding of measurement and branching out into number theory and analysis. It is remarkable that we are able to share so much of this with students in the elementary grades.

Euclidean Magnitudes

Book V of Euclid's *Elements* present a theory of ratio and proportion based on a theory of measurement. We will examine what Euclid said (or seems to have assumed) about measurement and the entities that are measured. It should become clear that the unifying themes in Common Core Standards for measurement parallel Euclid.

The vocabulary of the Euclidean theory of measurement includes the following terms:

1. A *magnitude* is something that has a continuous quantitative attribute. Things that have length, area, volume, mass, duration, and so on are magnitudes.
2. Magnitudes come in *kinds*. Lengths (i.e., things with length) form a kind. Areas (i.e., things with area) form a kind. SImilarly, masses form a kind, durations form a kind, etc.
3. Within any kind, we can:
   - *Compare.* Given two objects of a kind, we can tell if they are equal (with respect to their kind), and if not, we can tell which is larger.
   - *Add.* Given two objects of a kind, we can add them to make a larger thing of the same kind. For example, we can put things with length end-to-end. We can put two masses together, etc.
   - *Duplicate and multiply.* We can make copies of a thing—as many as we like, and we may add a given magnitude to itself over and over to form a multiple. If A is a magnitude an we add together 3 copies of A, we call the result $3A$. If we add together m copies of A, we call the result mA.
   - *Subtract.* A smaller magnitude may be removed from larger one of the same kind.
   - Addition is not sensitive to the order in which the parts are joined or assembled (i.e., it is associative and commutative). Moreover, addition of the same magnitude to two others preserves order. In other words, if A is less than B, then $A + C$ is less than $B + C$. The same is true of subtraction; if A is less than B, then $A - C$ is less than $B - C$. Also, if A is the same as B (with respect to some quantitative attribute) and the same magnitude is added to (or subtracted from) both then the resulting magnitudes are the same.

The Basic Measurement Process

The measurement process takes as input two magnitudes of the same kind playing different roles. One is the magnitude that *is measured* and the other is the magnitude that *does the measuring*. In the following, we shall call often call the former X and the latter U. X is measured by U. U is also called the *unit*.

The measurement process produces a number that summarizes the relationship between X and U. If X is equal to exactly m Us, that is, if $X = mU$, then we say, "The measure of X by U is m." In this case, it is also appropriate to refer to m as the *ratio of X to U*. If X is greater than mU

and less than nU, then we say, "The measure of X by U is between m and n," or "The ratio of X to U is between m and n."

Procedurally, the measurement of X by U may be performed in different but equivalent ways. One might add copies of U over and over, forming $2U$, $3U$, $4U$, etc., and compare the successive magnitudes to X to determine the largest that is less than X. This is in effect what happens when we measure with a ruler, for the ruler is a physical object on which the lengths U, $2U$, $3U$, etc. are represented and labeled. Another way to measure is to subtract copies of U over and over from X, counting as one goes, until it is impossible to subtract any more. For example, to find out how many cups of water there are in a container, we might start with a full container and then fill cups from it until we have used up all the water.

If X is not equal to a whole number of Us, then the measurement process still leads to a number, but the process by which the number is found is more complex. We will examine this in the next section.

Problems

1. Review the Common Core Standards for Measurement in Grades K-5, and identify where and how the notion a Euclidean magnitude appears and how the measurement process is conveyed.
2. Examine the units and activities concerning measurement in the Louisiana Comprehensive Curriculum for Grades K-5 (available at the Louisiana Department of Education web site). How do the Louisiana GLEs compare to the standards in the Common Core? What activities in this curriculum support the suggestions of the Common Core?

Section C. Mathematical Concepts Elaborated

Three Ways to Measure

There are three ways to use a unit U to measure an object X when X is not a multiple of U:

1) by dividing the unit,
2) by multiplying the thing measured,
3) by the Euclidean Algorithm (using remainders as new units to measure previous units).

1) Dividing the unit. Subtract as many whole copies of U as possible from X. Remember how many were subtracted. If there is some of X left over, divide U into equal parts (say eighths). Find out how many of these parts fit inside the remainder. If a second (smaller) remainder is obtained, divide U further, and find out how many of these fit inside the second remainder. Continue in this way until no remainder is left, or we achieve a measurement of accuracy sufficient to please us.

Example. To measure the width of the room (the object) in feet (the unit), lay a tape measure across the floor, with the end against one wall. If the width is not a whole number of feet, we count the inches between the last foot mark on the tape measure and the other wall. If there are not whole number of inches in this piece of tape, we count the quarter inches, etc.

Example. Decimals give us a variant of this. To measure X by U, subtract as many copies of U as possible from X, and record the number of subtractions. Call this "the whole-number part of the measure". If there is a remainder, divide U into 10 equal parts and find out how many of these fit in the remainder. Call this the "first decimal digit of the measure". If there is a second remainder, we divide the tenth part of U into 10 equal parts, and use one of these to count how many(1/100)ths of U remain. Etc.

The idea of dividing units can be reversed. If we want to measure a large thing, then using a multiple of the unit is reasonable. In the English system, a foot is 12 inches and a mile is 5280 feet. In the metric system larger and smaller units of length are related by powers of 10. A meter forms the basic unit of length. A decimeter is 1/10 of a meter; a centimeter is 1/100 of a meter; a millimeter is 1/1000 of a meter. Lengths longer than a meter are also named. A decameter is 10 meters; a hectometer is 100 meters and a kilometer is 1000 meters.

2) Multiplying the thing measured. Find out how many Us fit into X. Then find out how many Us fit into 2 Xs. Then find out how many Us fit into 3 Xs, etc. To record the data we acquire in this manner, we make a record of all the whole numbers m and n so that m Us fit inside of n Xs. We call this the set of lower pairs:

$$\mathrm{Lower}(X,U) := \{\,(n,m) \mid mU < nX\,\}.$$

Note that we have put n, the number of Xs being measured, in the first position. If the points (n,m) are depicted in the plane using a standard coordinate system, they will all lie under the line $y=\mu x$, where μ is the exact measure of X by U. For example, if $X=3U$, then the measure of X by U is 3, and we have $mU \leq nX$ if and only if $m \leq 3n$. We can also record the upper pairs:

$$\mathrm{Upper}(X,U) := \{\,(n,m) \mid mU > nX\,\}.$$

They lie *above* the line $y=\mu x$.

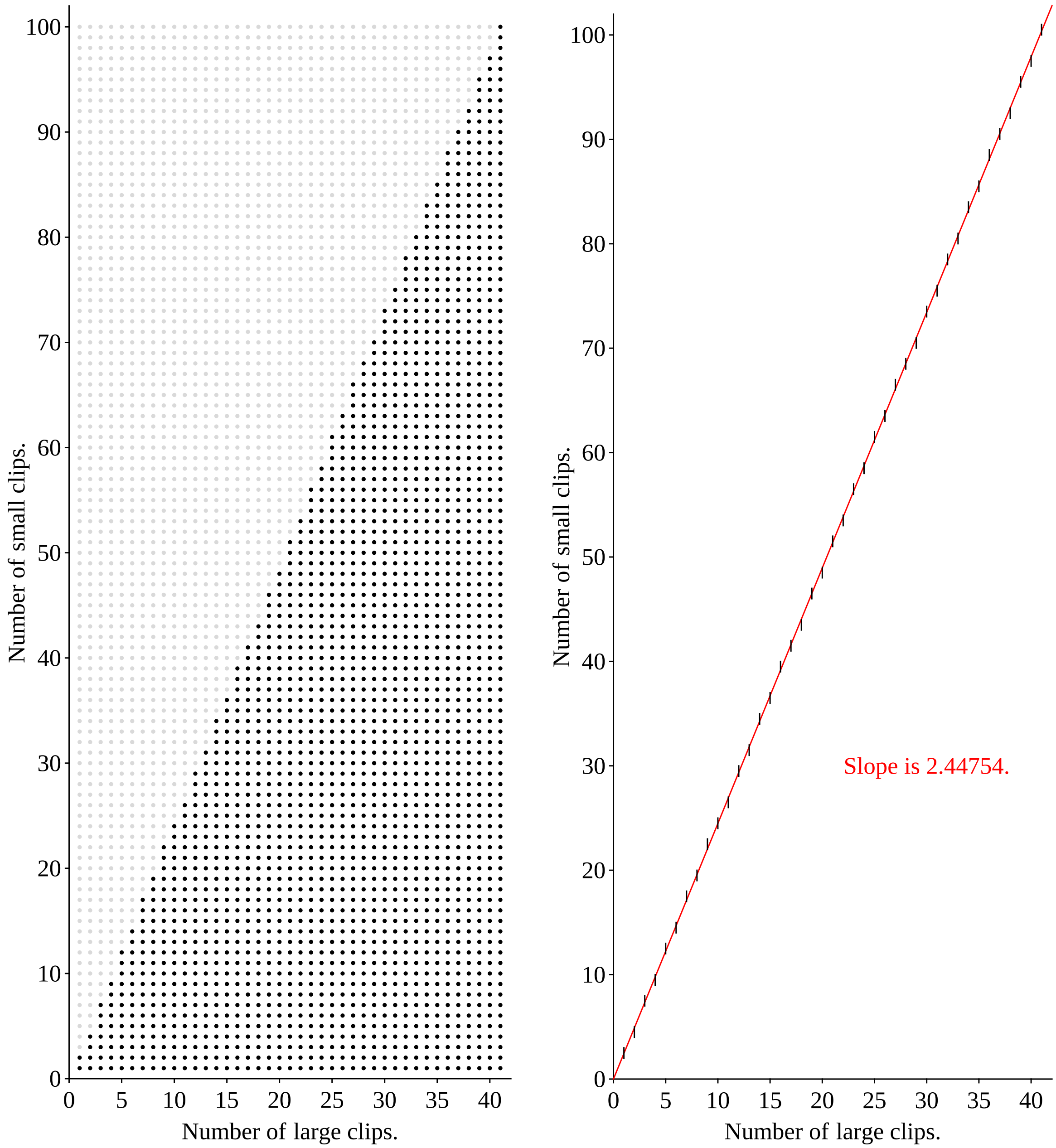
Problems

1. Suppose U is a centimeter and X is an inch. Using rulers labeled in both centimeters and inches, determine and graph lower and upper pairs.

2. Obtain several boxes of little paper clips and of big ones. Using a sensitive balance, weigh collections of paper clips—big ones and little ones—against one another. Graph them the lower and upper pairs.

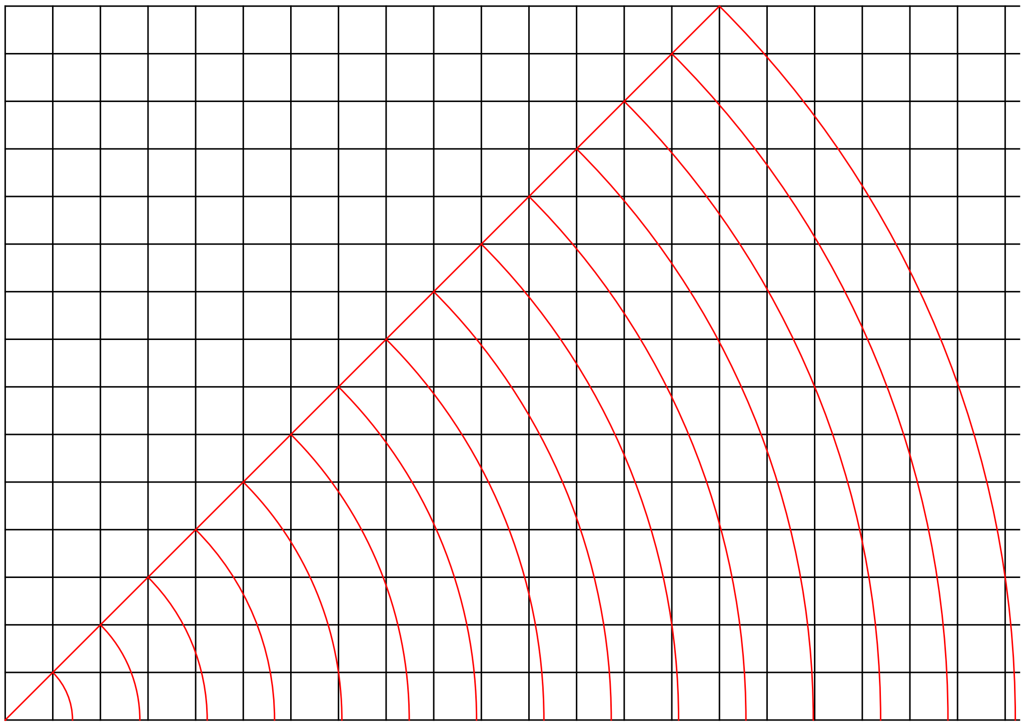
   Remarks. A small paper clip weighs just a bit more than 1/2 of a gram, and a large one weighs a little less than 5/4 of a gram. A balance that is sensitive to 1/100 of a gram can be made by driving pins partway through the center (18 inches) and ends (1 inch and 35 inches) of a wooden yardstick. Hang the yardstick from a pair of coat-hanger wire loops about the protruding ends of the center pin and hang hooks from the end pins to hold paperclips. Below is data from a real experiment with such a piece of equipment. After checking and adjusting the balance successive numbers of large clips were placed on one end, and small clips were hung from the other end until the balance tipped. This table shows the maximum number of small clips that fail to tip the balance against the corresponding number of big ones.

# large	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
# small	2	4	7	9	12	14	17	19	22	24	26	29	31	34	36
– – ––															
# large	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
# small	39	41	43	46	48	51	53	56	58	61	63	66	68	70	73
– – ––															
# large	31	32	33	34	35	36	37	38	39	40	41				
# small	75	78	80	83	85	88	90	92	95	97	100				



The data from balancing paper clips is graphed above. The picture on the left shows the elements of Lower(Big Paper Clip, Small Paper Clip) as dark dots and the elements of Upper(Big Paper Clip, Small Paper Clip) as gray dots. On the right, the vertical bars show upper and lower bounds for the exact value of the weight of x big paper clips, measured in units of small paper clips. The slope of the line is the best estimate of the weight of one large paper clip, measured with an average small paper clip as a unit.

3. Let U be the edge of a square and let X be the diagonal. Use the picture below to find the elements of $\text{Lower}(X, U)$ and of $\text{Upper}(X, U)$ that have $n \leq 15$.



3) The Euclidean Algorithm. Here, we use remainders to create new units. If X and U are magnitudes and $X > U$, then we subtract U from X as many times as possible. Call the number of times subtracted b_0 and call the remainder U_1. Then $U > U_1$. Subtract U_1 from U as many times as possible. Call the number of times subtracted b_1 and call the remainder U_2. Continue:

$$\begin{array}{lclclcll} X & = & b_0 & \cdot & U & + & U_1, & U_1 < U \\ U & = & b_1 & \cdot & U_1 & + & U_2, & U_2 < U_1 \\ U_1 & = & b_2 & \cdot & U_2 & + & U_3, & U_3 < U_2 \\ U_2 & = & b_3 & \cdot & U_3 & + & U_4, & U_4 < U_3 \\ & \vdots & & & & & & \end{array}$$

One continues as long as new remainders are produced. This results in a sequence

$$E(X,U) := (b_0, b_1, b_2, \ldots).$$

The process of building this sequence is called the *Euclidean Algorithm*. For example, if we apply the Euclidean Algorithm to 27 and 10, we get:

$$\begin{array}{rcrcrcr} 27 & = & 2 & \cdot & 10 & + & 7 \\ 10 & = & 1 & \cdot & 7 & + & 3 \\ 7 & = & 2 & \cdot & 3 & + & 1 \\ 3 & = & 3 & \cdot & 1 & + & 0 \end{array}$$

Problem. Find $E(31, 17)$.

Let us apply the Euclidean Algorithm to the diagonal and the edge of a square. One way to do this is by using the decimal expansion of $\sqrt{2}$. Here, we will let $X = \sqrt{2} = 1.414213562373095\ldots$, and $U = 1$.

$$\begin{array}{rclcrcl}
\sqrt{2} & = & 1 & \cdot & 1 & + & 0.414213562373095\ldots \\
 & = & 2 & \cdot & 0.414213562373095\ldots & + & 0.171572875253809\ldots \\
0.4142135623730950\ldots & = & 2 & \cdot & 0.171572875253809\ldots & + & 0.071067811865475\ldots \\
0.1715728752538099\ldots & = & 2 & \cdot & 0.071067811865475\ldots & + & 0.029437251522859\ldots \\
0.0710678118654752\ldots & = & 2 & \cdot & 0.029437251522859\ldots & + & 0.012193308819756\ldots \\
0.0294372515228594\ldots & = & 2 & \cdot & 0.012193308819756\ldots & + & 0.005050633883346\ldots \\
0.0121933088197564\ldots & = & 2 & \cdot & 0.005050633883346\ldots & + & 0.002092041053063\ldots \\
0.0050506338833465\ldots & = & 2 & \cdot & 0.002092041053063\ldots & + & 0.000866551777220\ldots \\
0.0020920410530632\ldots & = & 2 & \cdot & 0.000866551777220\ldots & + & 0.000358937498623\ldots \\
0.0008665517772200\ldots & = & 2 & \cdot & 0.000358937498623\ldots & + & 0.000148676779973\ldots \\
0.0003589374986231\ldots & = & 2 & \cdot & 0.000148676779973\ldots & + & 0.000061583938675\ldots \\
0.0001486767799739\ldots & = & 2 & \cdot & 0.000061583938675\ldots & + & 0.000025508902624\ldots \\
0.0000615839386752\ldots & = & 2 & \cdot & 0.000025508902624\ldots & + & 0.000010566133428\ldots
\end{array}$$

The fact that all the b_is except the first one are equal to 2 is notable. The Greeks knew that the 2s went on forever 2400 years ago; the fact is mentioned in Plato's work. They figured this out with geometry. The Greeks did not know, beyond the first few lines of the following, what would result if the Euclidean Algorithm were applied to π:

$$\begin{array}{rclcrcl}
\pi & = & 3 & \cdot & 1 & + & 0.14159265358979\ldots \\
1 & = & 7 & \cdot & 0.14159265358979\ldots & + & 0.00885142487144\ldots \\
0.1415926535897932\ldots & = & 15 & \cdot & 0.00885142487144\ldots & + & 0.00882128051808\ldots \\
0.0088514248714473\ldots & = & 1 & \cdot & 0.00882128051808\ldots & + & 0.00003014435336\ldots \\
0.0088212805180832\ldots & = & 292 & \cdot & 0.00003014435336\ldots & + & 0.00001912933578\ldots \\
0.0000301443533640\ldots & = & 1 & \cdot & 0.00001912933578\ldots & + & 0.00001101501758\ldots
\end{array}$$

No pattern grabs your attention, and in fact no simple pattern is known. The first 36 b_is that arise when we measure π by 1 are:

$$3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, 1, 84, 2, 1, 1, 15, 3, 13, 1, 4, 2, 6, 6, 99, 1, 2, \ldots$$

Continued Fractions

An alternative way of expressing the Euclidean Algorithm is by writing the so-called *continued fraction expansion*. Let's illustrate with 31/17. First, recall the computation that you made showing how 31 was measured by 17:

$$\begin{array}{rcrcrcr}
31 & = & 1 & \cdot & 17 & + & 14 \\
17 & = & 1 & \cdot & 14 & + & 3 \\
14 & = & 4 & \cdot & 3 & + & 2 \\
3 & = & 1 & \cdot & 2 & + & 1 \\
2 & = & 2 & \cdot & 1 & + & 0
\end{array}$$

Now observe the following:

$$\begin{aligned}
\frac{31}{17} &= \mathbf{1} + \frac{14}{17} \\
&= \mathbf{1} + \frac{1}{17/14} \\
&= \mathbf{1} + \frac{1}{\mathbf{1} + \frac{3}{14}} \\
&= \mathbf{1} + \frac{1}{\mathbf{1} + \frac{1}{14/3}} \\
&= \mathbf{1} + \frac{1}{\mathbf{1} + \frac{1}{\mathbf{4} + \frac{2}{3}}} \\
&= \mathbf{1} + \frac{1}{\mathbf{1} + \frac{1}{\mathbf{4} + \frac{1}{3/2}}} \\
&= \mathbf{1} + \frac{1}{\mathbf{1} + \frac{1}{\mathbf{4} + \frac{1}{\mathbf{1} + \frac{1}{\mathbf{2}}}}}
\end{aligned} \tag{$*$}$$

Problems I. (Comment for mathematicians: The following series of problems is still a work in progress. The goal is to provide some experience on which to begin building ideas about rational approximation. The problem of convergence is addressed elsewhere.)

1. What procedure was followed in the computation ($*$)? Why do the numbers in bold turn out to be the same as the numbers in the Euclidean Algorithm?

2. Consider the (improper) fractions with denominator less than 17. Among those less than 31/17, which is the *closest* to 31/17? Among those larger than 31/17, which is the *closest* to 31/17? (Use the chart I have attached.) Then, simplify the following:
   a. $1 + \frac{1}{1+1}$
   b. $1 + \frac{1}{1+\frac{1}{4}}$
   c. $1 + \frac{1}{1+\frac{1}{4+1}}$
   d. $1 + \frac{1}{1+\frac{1}{4+\frac{1}{1+\frac{1}{1}}}}$

3. How do the following numbers compare to $\sqrt{2}$?
   a. $1 + \frac{1}{2}$
   b. $1 + \frac{1}{2+\frac{1}{2}}$
   c. $1 + \frac{1}{2+\frac{1}{2+\frac{1}{2}}}$
   d. $1 + \frac{1}{2+\frac{1}{2+\frac{1}{2+\frac{1}{2}}}}$

Problems II. (These problems are also a work in progress. The goal is to work toward showing that a periodic continued fraction represents a quadratic irrational.)

1. If we tried to write the continued fraction for $1 + \sqrt{2}$ is would look like this:

$$2 + \frac{1}{2 + \frac{1}{2+\frac{1}{2+\frac{1}{2+\cdots}}}}.$$

Here's a tricky way show that this must equal $1+\sqrt{2}$. First, set

$$X = 2 + \cfrac{1}{2 + \frac{1}{2+\frac{1}{2+\frac{1}{2+\cdots}}}}.$$

Then

$$X = 2 + \cfrac{1}{\left(2 + \frac{1}{2+\frac{1}{2+\frac{1}{2+\cdots}}}\right)} = 2 + \frac{1}{X}.$$

Thus,

$$X = 2 + \frac{1}{X}.$$

Now, solve for X.

2. By the same reasoning, what should the following equal?

$$1 + \cfrac{1}{1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+\cdots}}}}.$$

3. What is this?

$$b + \cfrac{1}{b + \frac{1}{b+\frac{1}{b+\frac{1}{b+\cdots}}}}.$$

4. What is this?

$$2 + \cfrac{1}{7 + \frac{2}{7+\frac{2}{7+\frac{1}{2+\cdots}}}}.$$

Problems III. (The following problems head toward the converse: the partial fraction representation of a quadratic irrational is periodic.)

1. Find $E(\sqrt{2},1)$ without using the decimal expansion, but instead the fact that

$$\frac{1}{\sqrt{2}-1} = \sqrt{2}+1.$$

2. Find $E(\sqrt{3},1)$, using algebraic properties of $\sqrt{3}$.
3. Find $E(\sqrt{5},1)$, $E(\sqrt{6},1)$ and $E(\sqrt{7},1)$.
4. Find the following: $E(\sqrt{n^2+1},1)$, $E(\sqrt{n^2+n},1)$ and $E(\sqrt{n^2-1},1)$.

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