Totally ordered commutative monoids

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This is a draft. Section 7 is incomplete, and section 9 is not included.

Abstract. *A totally ordered monoid—or tomonoid, for short—is a commutative semigroup with identity S equipped with a total order $\leq_S$ that is translation invariant, i.e., that satisfies: $\forall x, y, z \in S \quad x \leq_S y \;\Rightarrow\; x + z \leq_S y + z$. We call a tomonoid that is a quotient of some totally ordered free commutative monoid* formally integral. *Our most significant results concern characterizations of this condition by means of constructions in the lattice $\mathbf{Z}^n$ that are reminiscent of the geometric interpretation of the Buchberger algorithm that occurs in integer programming. In particular, we show that every two-generator tomonoid is formally integral. In addition, we give several (new) examples of tomonoids that are not formally integral, we present results on the structure of nil tomonoids and we show how a valuation-theoretic construction due to Hion reveals relationships between formally integral tomonoids and ordered commutative rings satisfying a condition introduced by Henriksen and Isbell.*

0. Introduction.

In his 1976 survey of ordered semigroups [G], E. Ya. Gabovich identified several general research problems. The present work contains results that respond directly to at least three of these. First, Gabovich asked explicitly for criteria for formal integrality (and related properties in possibly non-commutative varieties of semigroups). Second, he posed the general problem of developing structure theories for classes of ordered semigroups. Third, he singled out Hion's work as a potentially useful way of describing the structure of totally ordered rings. We now comment on each of these topics in more detail.

For clarity, we define a few terms that will be used in the introduction. A monoid is a set with an associative binary operation and an identity element. All monoids in this paper are commutative. The concept of a tomonoid is defined in the abstract. Let S be a tomonoid. We say S is *positive* if $0_S \leq_S x$ for all $x \in S$. If S is positive, we say S is *archimedean* if for any $x, y \in S \setminus \{0_S\}$ there is a positive integer n such that $x \leq_S ny$. When we speak of a *quotient* of S, we always intend a congruence relation θ with convex classes, so the natural surjection $S \twoheadrightarrow S/\theta$ induces a translation-invariant total order on S/θ. We view S/θ as a tomonoid with this order. When we speak of a *sub-tomonoid* of S, we intend a sub-monoid with induced order.

Problem 6 of [G] reads: *In a variety (of semigroups) whose free semigroups are orderable, find necessary and sufficient conditions for a totally ordered semigroup to be a quotient of some totally ordered free semigroup*. All free commutative monoids are orderable—indeed the positive orders on finitely generated free commutative monoids are the so-called "term orders" that figure prominently in Buchberger's algorithm. Thus the description of formally integral tomonoids is an instance of Gabovich's problem. We present a useful characterization of formal integrality in Proposition 4.4, and as a corollary (4.5) we find that every cancellation tomonoid is formally integral (thus, answering another question from [G]). Much deeper applications of 4.4 include Theorem 4.8, stating that any two-generator tomonoid is formally integral and Theorem 6.1, stating that a positive tomonoid whose generators are 'spaced sufficiently far apart' (in a sense made precise in the statement of the theorem) is formally integral. The proofs of 4.8 and 6.1 were inspired by Thomas's geometric interpretation of the Buchberger algorithm [T].

Examples of tomonoids that are not formally integral have appeared previously (see [HI], [I1], [I2] and [GIR]), but in all cases these examples have required at least 4 generators. In §5, we present examples of 3-generator tomonoids that are not formally integral, including one example that has 9 elements. (Whipple has recently shown that all positive tomonoids with 8 or fewer elements are formally integral, [W].) In §9,we describe a *Mathematica* program we wrote that searches for 3-generator positive tomonoids that are not formally integral and determines isomorphism types. Using it, found nearly two hundred (non-isomorphic) examples. Typically, these are not easy to come by. For example, in a collection of 8000 candidates that met conditions required by Theorem 6.1, we found that less than 1% failed to be formally integral. The program uses a convex hull computation and depends upon a good deal of the theory developed in the present paper. It is available by email on request from the senior author at *madden@math.lsu.edu*.

We turn to the second area where the present work responds to Gabovich's program. His review devotes considerable attention to problems arising in the structure theory of ordered semigroups. Every totally ordered semigroup decomposes into a disjoint union of archimedean subsemigroups—the archimedean classes. (See [G], page 179 for a discussion, which involves some delicate definitions.) Thus, solutions to the following problems would provide a basis for a complete structure theory:

- *First structure problem*: Determine the structure of an arbitrary archimedean totally ordered semigroup.
- *Second structure problem*: Determine the ways in which a given chain of archimedean totally ordered semigroups can be assembled to form a totally ordered semigroup having the elements of the chain as its archimedean classes.

At present, we are nowhere near a solution of the more basic first problem, even in the commutative case. Therefore, it is reasonable to hope to make inroads by developing structure theories for classes of totally ordered semigroups that are defined by properties stronger than archimedean. Our results have a bearing on one such class. In the terminology of [G], an additive semigroup S is said to be *nil* if it has an absorbing element ∞ and for each element $x \in S$ there is $n \in \mathbf{N}$ such that $nx = \infty$. It is clear that a positive totally ordered nilsemigroup is archimedean. Problem 16 of [G] asks for *a description of all totally ordered nilsemigroups ..., (in the first instance, the finite ones).*

In the present paper, a tomonoid S is called nil if it is positive and $S \setminus \{0\}$ is nil in the sense of [G]. We show (7.2) that every finite formally integral nil tomonoid is a quotient of a sub-tomonoid of $\mathbf{N}$. This reduces the structure problem for this class to the problem of classifying convex congruences on sub-tomonoids of $\mathbf{N}$. This would appear to be a tractable problem, especially in view of Grillet's work [Gr] on congruences on commutative semigroups. As for finite nil tomonoids that fail to be formally integral, proposition 4.7 provides a reasonable starting point; see the remarks at the end of §7. We plan to take up these problems in a future paper.

During the work leading to the present paper, we considered but did not answer what might be the most basic problem relating to the second structure problem:

Problem 1. *If (S is positive and) each archimedean class of S is formally integral, does it follow that S is formally integral?*

We now make a few remarks on a property which received a lot of attention in the work of Clifford [C] and which was featured in Fuchs's textbook [F] and in the review [HL]. We say S is *naturally ordered* if it is positive and $x \leq_S y$ implies that $y = x + z$ for some $z \in S$. The theorem of Clifford, Hölder and Huntington (see [C] or [HL]) implies that every naturally ordered positive archimedean tomonoid is order-isomorphic to a sub-tomonoid of one of the following tomonoids:

$$\mathbf{R}_{\geq 0}\ , \quad \mathbf{R}_{\geq 0}/[1,\infty) \quad \text{or} \quad \mathbf{R}_{\geq 0}/(1,\infty).$$

Here, $\mathbf{R}_{\geq 0}$ denotes the non-negative reals under addition, and the latter two tomonoids are quotients of $\mathbf{R}_{\geq 0}$ each with exactly one non-singleton congruence class, $[1,\infty)$ and $(1,\infty)$, respectively. Thus, we have the implication

$$\text{positive, naturally ordered and archimedean} \Rightarrow \text{quotient of a sub-tomonoid of } \mathbf{R}_{\geq 0}.$$

This implication is obviously not reversible. For example, the sub-tomonoid of $\mathbf{R}_{\geq 0}$ generated by 2 and 3 is not naturally ordered. By 4.2 and 4.5, we have the implication

$$\text{quotient of a sub-tomonoid of } \mathbf{R}_{\geq 0} \Rightarrow \text{formally integral and archimedean}.$$

These observations raise a couple of questions that we have not attempted to answer:

Problem 2. *Is every positive formally integral archimedean tomonoid a quotient of a sub-tomonoid of $\mathbf{R}_{\geq 0}$?*

A special case of this—applying to finite formally integral nil tomonoids—is true by 7.2. Relevant to Problem 2 is the work of Alimov, who found necessary and sufficient conditions for a tomonoid to be isomorphic to a sub-tomonoid of $\mathbf{R}_{\geq 0}$. His work is discussed in [F], [HI], [Sa]. We also pose the following, which is related to Problem 1:

Problem 3. *Is every naturally ordered tomonoid formally integral?*

The third way in which our work is related Gabovich's program concerns applications to ordered rings. This theme originates in work of Hion from the 1950's, in which he worked out a kind of generalized valuation theory for arbitrary ordered rings; see [Hi], [G]. A Hion valuation is a map from a totally ordered rings to its tomonoid of (additive) archimedean classes. Connections between ordered semigroups and ordered rings were also considered by Henriksen and Isbell in the early 1960's; see [HI]. They introduced a generalization of the concept of a formally real field that is meaningful for lattice-ordered commutative rings. They showed that a totally ordered ring is formally real (in their extended sense) if and only if it is a quotient of a totally ordered domain. In producing examples of non-formally real rings, they used the fact that a totally ordered monoid algebra over a non-formally integral tomonoid is never formally real. We describe all this in more detail in §8.

In recent years, tomonoid structures on $\mathbf{N}^n$—*i.e.*, term orders—have played an important role in connection with Gröbner bases, toric varieties and integer programming; see [St]. We hope that readers familiar with this will find connections with what we report here. For example, can techniques used in [T] lead to an intersting structure theory for finite tomonoids, or help in solving the problem of characterizing the finite monoids that admit non-formally-integral total orders? Is the (fairly obvious) generalization Gröbner bases to monoid algebras over positive tomonoids of any use or significance?

As to the contents of this paper, §1 introduces notation and §2 reviews well-known results on ordered groups. Each total order on an abelian group is completely determined by its positive cone. For monoids, it is impossible to associate cones with orders. The nearest analogue is the difference set. We define difference sets in §3 and prove some technical lemmas about them. In §4, we introduce the formally integral tomonoids and use difference sets to prove a number of propositions about them, including Theorem 4.8, which states that every 2-generator tomonoid is formally integral. §5 discusses examples of 3-generator tomonoids that are not formally integral. In §6, we use the same method of proof as in 4.8 to obtain a more general theorem that we expect to be useful for proving theorems about Hion tomonoids of a totally ordered rings. §7 contains some results on nil tomonoids. §8 discusses connections with ordered rings, and §9 describes an algorithm we used to search for examples of non-formally-integral tomonoids with 3 generators.

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1. Notation, terminology and preliminaries.

This section sets up terminology and concludes with a couple of useful lemmas (which are certainly not original with us, but for which we know no reference). We consider partially ordered monoids, even though they are not considered in the rest of this paper, because they are likely to play a role in future work.

A monoid is a set with an associative binary operation having an identity. All monoids considered here are commutative, and the term "monoid" always means commutative monoid. $\mathbf{N}$, $\mathbf{Z}$ and $\mathbf{Q}$ denote, respectively, the non-negative integers, the integers and the rationals. We will generally use additive notation. Let S be a monoid. The operation in S is denoted $+$; the identity element is denoted 0 (or 0_S); the n-fold sum $s + s + \cdots + s$ is denoted ns. We say that S is generated by a subset $E \subseteq S$ if every element of S is of the form $s = n_1e_1 + \cdots n_ke_k$ for some $n_1, \ldots n_k \in \mathbf{N}$ and $e_1, \ldots e_k \in E$.

We shall make frequent use of free monoids, so we review some basic facts concerning them. Let I be a set. The *free monoid on I* is denoted $F_{_N}(I)$. It is the monoid of all functions $x : I \to \mathbf{N}$ such that $x(i) \neq 0$ for at most finitely many $i \in I$. Thus, if I is finite—as it will be in most applications considered here—$F_N(I) = \mathbf{N}^I$. For $i \in I$, let $\epsilon_i : I \to \mathbf{N}$ denote the function defined by

$$\epsilon_i(j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

These generate $F_N(I)$. $F_N(I)$ has the following universal mapping property: for any monoid S and any set map $f : I \to S$, there is a unique monoid homomorphism $\overline{f} : F_N(I) \to S$ such that $f(i) = \overline{f}(\epsilon_i)$ for all $i \in I$.

$F_Z(I)$ and $F_Q(I)$ denote, respectively, the free abelian group and the free $\mathbf{Q}$-vector-space on I. The description of these objects is analogous to the description of $F_N(I)$, but with $\mathbf{Z}$ or $\mathbf{Q}$ in place of $\mathbf{N}$. We have natural containments $F_N(I) \subseteq F_Z(I) \subseteq F_Q(I)$.

Now we turn to orderings. Let $\leq_S$ be a partial order relation on S. We say that $\leq_S$ is *translation-invariant* if

$$\forall\, s, t, u \in S \quad s \leq_S t \;\Rightarrow s + u \leq_S t + u. \tag{TI}$$

We say that $\leq_S$ is *a-saturated* if

$$\forall\, s, t, u \in S \quad s + u <_S t + u \;\Rightarrow s <_S t.$$

We say that $\leq_S$ is *m-saturated* if for all $s, t \in S$ and all positive integers n, $ns <_S nt$ implies $s <_S t$. It is easy to see that every translation-invariant *total* order is both a-saturated and m-saturated.

A monoid equipped with a translation-invariant partial order is called a *pomonoid*. A monoid equipped with a translation-invariant total order is called a *tomonoid*. We say that $\leq_S$ is *positive*—or that $(S, \leq_S)$ is a *positive pomonoid*—if for all $s \in S$, $0_S \leq_S s$. Similarly, we say that $\leq_S$ is *negative* if for all $s \in S$, $s \leq_S 0_S$. Suppose S and T are pomonoids. A function $\phi : S \to T$ is called a homomorphism if $\phi(s+t) = \phi(s) + \phi(t)$ for all $s, t \in S$. If in addition $s \leq_S t$ implies $\phi(s) \leq_T \phi(t)$ for all $s, t \in S$, then we call ϕ a *pomonoid morphism*, or, if S and T are totally ordered, a *tomonoid morphism*.

It is often useful to consider all the ways a given monoid may be ordered. If $\leq_S$ is a translation-invariant partial order, then we call $\leq_S$ itself a *TIPO*. If $0_S \leq_S s$ for all $s \in S$, then we speak of *positive TIPO*. Similarly we may speak of an a-saturated or m-saturated TIPO. An intersection (in $S \times S$) of any family of TIPO's is a TIPO. Thus, if a relation $R \subseteq S \times S$ is contained in a TIPO, there is a weakest TIPO that extends R, and it is called the TIPO *generated by* R. A TIPO that is a total order will be called it a *TITO*. In this paper, we have little more to say about general TIPO's; we have mentioned them only in conjunction with our review of facts about TITO's. An answer to the following question, however, would be extremely useful in extending the results of subsequent sections to pomonoids.

Problem 4. *Let S be a monoid. Characterize those TIPO's on S that are intersections of TITO's.*

Clearly, such a TIPO must be a-m-saturated, but we do not know if every a-m-saturated TIPO is an intersection of TITO's.

We now discuss a particularly simple kind of congruence that we use frequently. Suppose that $(S, \le_S)$ is a tomonoid. A subset $K \subseteq S$ is called an upper interval if $s \in K$ and $t >_S s$ implies $t \in K$. If K is an upper interval and $S + K \subseteq K$, then S/K denotes the tomonoid whose elements are the $x \in S$ such that $x \notin K$ together with one more element, denoted ∞. The operation is the same as in S, except that if $s_1 + s_2 \in K$ (in S), then we put $s_1 + s_2 = \infty$ (in S/K), and for any s, $s + \infty = \infty$. The order is such that ∞ greater than every other element, and otherwise is the same as in S. There is a surjective tomonoid monomorphism $\pi_K : S \to S/K$ that sends every element in K to ∞ and is the identity elsewhere. If $w \in S$, S/w is an abbreviation for $S/\{\, s \in S \mid w \le_S s \,\}$ (if it is defined), and $\pi_w : S \to S/w$ denotes the corresponding quotient map.

The following proposition and its corollaries are useful elementary facts.

Proposition 1.1. *Any sequence of elements in a finitely generated positive pomonoid contains a non-decreasing subsequence.*

Proof. Suppose S is generated by $e_1, \ldots, e_n$. Suppose $\{s_i\}$ is a sequence of elements of S. Choose a sequence $\{\nu_i\}$ of elements of $\mathbf{N}^n$ such that $s_i = \nu_{i1}e_1 + \cdots \nu_{in}e_n$. Take a subsequence $\{\nu_i^{(1)}\} \subseteq \{\nu_i\}$ so that $\{\nu_{i1}^{(1)}\}$ is non-decreasing. Choose a subsequence $\{\nu_i^{(2)}\} \subseteq \{\nu_i^{(1)}\}$ so that $\{\nu_{i2}^{(2)}\}$ is non-decreasing. Continuing in this manner, we obtain a sequence $\{\nu_i^{(n)}\}$ so that $\{\nu_{ij}^{(n)}\}_{i=1}^{\infty}$ is non-decreasing for all $j = 1, \ldots, n$. Then $\{\nu_{i1}^{(n)}e_1 + \cdots \nu_{in}^{(n)}e_n\}_{i=1}^{\infty}$ a non-decreasing subsequence of $\{s_i\}$. ■

Corollary 1.2. *A finitely-generated positive pomonoid satisfies the descending chain condition. A finitely-generated positive tomonoid is well-ordered, i.e., every subset has a least element.* ■

Let S be a tomonoid. A set E of generators for S is *minimal* if no subset of E generates S.

Corollary 1.3. *Every finitely-generated positive tomonoid has a unique minimal set of generators.*

Proof. Let $s_1 <_S \cdots <_S s_m$ and $s'_1 <_S \cdots <_S s'_n$ be two minimal generating sets for S. Clearly, $s_1 = s'_1$ is the smallest non-zero element of S. Suppose we have found $s_i = s'_i$ for $i = 1, \ldots, k$. Then, $s_{k+1} = s'_{k+1}$ must be the least element of S that is not in $\langle s_1, \ldots s_k \rangle$. ■

The proof shows that the minimal generating set is minimal in two senses: it has the fewest number of elements *and* the elements are as small (in the relevant TITO) as possible.

Many of the definitions we have made (and the results to follow) extend to (commutaive) semigroups S equipped with orders satisfying (TI). In making generalizations care is needed, unless S is positive (*i.e./*, $s + t \ge s$ for all $s, t \in S$). If S is positive, then we can obviously adjoin an identity 0 such that $0 \le s$ for all $s \in S$. Because of this, in considering positive semigroups, the presence or absence of an identity makes little difference.

2. Orders on abelian groups.

This section treats two topics. First, we review the well-known correspondence between pointed convex cones and TIPOs in a **Q**-vector space. After this, we present a proposition that solves the problem of classifying all TITO's on a free commutative monoid by reducing it to the problem of classifying all TITO's on a a **Q**-vector space of the same rank.

Let V be a **Q**-vector-space. A non-empty subset $C \subset V$ is called a pointed convex cone or a *P-cone* if C contains no subspace and $\lambda v + \mu w \in C$ whenever λ and μ are nonnegative rationals and v and w are in C. If $C \subset V$ is a P-cone, then the relation $\leq_C$ on V defined by

$$v \leq_C w \;\Leftrightarrow w - v \in C$$

is an m-saturated TIPO. Conversely, if $\leq$ is an m-saturated TIPO on V then $P(\leq) := \{\, v \in V \mid 0_V \leq v \,\}$ is a P-cone. Moreover, $P(\leq_{P_0}) = P_0$ for any P-cone P_0, and $\leq_{P(\leq_0)} = \leq_0$ for any m-saturated TIPO $\leq_0$.

If C is a P-cone and $C \cup -C = V$, we call C a *T-cone*. $\leq_C$ is a TITO if and only C is a T-cone. It can be shown that a T-cone is just a maximal P-cone and that every P-cone is equal to the intersection of all the T-cones that contain it.

Let A be a torsion-free abelian group, let $V := \mathbf{Q} \otimes A$ be the divisible hull of A. We call $C' \subseteq A$ a P-cone (respectively, a T-cone) in A if $C' = C \cap A$ for some P-cone (respectively, T-cone) $C \subseteq V$. As with vector spaces, there is a one-to-one correspondence between P-cones in A and m-saturated TIPO's, with T-cones corresponding to TITO's.

The next proposition states that any TITO on $F_N(I)$ is the restriction to $F_N(I)$ of a TITO on $F_Q(I)$. This fairly well-known fact is not difficult to prove, but because the proof illustrates notation that will be useful later, we include it. We use the componentwise lattice order on $F_Z(I)$, which is denoted $\leq_\ell$ and is defined by

$$x \leq_\ell y \;\Leftrightarrow\; \forall i \in I\, x(i) \leq y(i).$$

For any $x, y \in F_Z(I)$, $x \vee y$, $x \wedge y$, x^+, and x^- have the following meanings:

- $\big(x \vee y\big)(i) := \max\{x(i), y(i)\}$,
- $\big(x \wedge y\big)(i) := \min\{x(i), y(i)\}$,
- $x^+ := x \vee 0$ and
- $x^- := (-x) \vee 0$.

Obviously:

$$0 \leq x^+ \,, \quad 0 \leq x^- \,, \quad \text{and} \quad x = x^+ - x^- .$$

Proposition 2.1. *Let I be a set. Any TITO on $F_N(I)$ has a unique extension to a TITO on $F_Z(I)$ and a unique extension to a TITO on $F_Q(I)$.*

Proof. Let $\leq_0$ be a TITO on $F_N(I)$. For any $x, y \in F_Z(I)$, $(x \wedge y)^-$ is the least element z in the order $\leq_\ell$ such that $x + z$ and $y + z$ are both in $F_N(I)$. Let $\leq_1$ be the relation on $F_Z(I)$ defined by

$$x \leq_1 y \;\Leftrightarrow\; x + (x \wedge y)^- \leq_0 y + (x \wedge y)^- .$$

It is clear that any TITO on $F_Z(I)$ that restricts to $\leq_0$ on $F_N(I)$ must satisfy this condition, and is uniquely determined by it. So, we only need to show that $\leq_1$ is a TITO on $F_Z(I)$.

It is obvious that $\leq_1$ is reflexive and anti-symmetric, and that for any pair of elements $x, y \in F_Z(I)$, either $x \leq_1 y$ or $y \leq_1 x$. It remains to verify that $\leq_1$ is transitive and translation-invariant. Note that for all $x, y \in F_Z(I)$,

$$x \leq_1 y \;\Leftrightarrow\; \text{for all } w \in F_N(I),\ (x \wedge y)^- \leq_\ell w \text{ implies } \ x + w \leq_0 y + w. \qquad (\dagger)$$

Suppose that $x \leq_1 y$ and $y \leq_1 z$. Pick $w \in F_N(I)$ such that $(x \wedge y)^- \vee (y \wedge z)^- \leq_\ell w$. By (†), $x+w \leq_0 y+w$ and $y+w \leq_0 z+w$. By transitivity of $\leq_0$, $x+w \leq_0 z+w$, and by (†) again, $x \leq_1 y$. Suppose that $x \leq_1 y$ and $z \in F_Z(I)$. Pick $w \in F_N(I)$ such that $(x \wedge y)^- \leq_\ell z+w$ and $((x+z) \wedge (y+z))^- \leq_\ell w$. Then by (†), $(x+z)+w = x+(z+w) \leq_0 y+(z+w) = (y+z)+w$, and by (†) again, $x + z \leq_1 y + z$. This completes the demonstration that $\leq_1$ is a TITO.

Next, let $\leq_2$ be the relation on $F_Q(I)$ defined by

$$x \leq_2 y \;\Leftrightarrow\; \exists k \in \mathbf{N} \setminus \{0\} \ \text{ such that } \ kx, ky \in F_Z(I); \&\ kx \leq_1 ky.$$

We leave it to the reader to perform the straightforward verification that this is a TITO on $F_Q(I)$ and that it is the only one that restricts to $\leq_1$. ▪

Any homomorphism $\phi : F_N(I) \to F_N(J)$ has a unique extension to a homomorphism $\overline{\phi} : F_Z(I) \to F_Z(J)$. If σ and τ are TITO's on $F_N(I)$ and $F_N(J)$, respectively, and $\overline{\sigma}$ and $\overline{\tau}$ are their extensions to $F_Z(I)$ and $F_Z(J)$, then $\overline{\phi}$ is a tomonoid morphism with respect to $\overline{\sigma}$ and $\overline{\tau}$. For if $x \leq y$ in $F_Z(I)$, then for some $z \in F_N(I)$, $x + z$ and $y + z$ are in $F_N(I)$. Then $\phi(x + z) \leq \phi(y + z)$ in $F_N(J)$, so $\overline{\phi}(x) \leq \overline{\phi}(y)$ in $F_Z(J)$.

3. Difference sets and cones.

In contrast to the free monoid case, the TITOs on a general monoid S are not in one-to-one correspondence with a set of cones in a vector space. Nonetheless, given a surjection $F_N(I) \twoheadrightarrow S$, we can associate with each TITO $\leq_S$ on S a cone in $F_Q(I)$, and this cone yields a lot of information $\leq_S$. Even more information about $\leq_S$ is recorded in the difference set of $\leq_S$, which we define and discuss below. Difference sets represent an attempt to extend the geometric picture of orderings on groups that we discussed in the previous section to the case of monoids; we think the results we obtain in §4 and §6 indicate that this attempt has been moderately successful.

Suppose that

$$\phi : F_N(I) \to S$$

is a monoid homomorphism. If $\leq_S$ is a TIPO on S and $<_S$ is the associated strict order, we let

$$D(\phi, <_S) := \{\, z \in F_Z(I) \mid \ z = y - x \text{ for some } x, y \in F_N(I) \text{ such that } \phi(x) <_S \phi(y) \,\},$$

and we call $D(\phi, <_S)$ the *difference set of* $<_S$ *by* ϕ. If the order on S is understood, we write $D(\phi)$ as shorthand for $D(\phi, <_S)$. The cone in $F_Q(I)$ generated by $D(\phi)$ is denoted $C(\phi)$. Note that $0_{F_N(I)}$ is *never* in $D(\phi)$ (but is *always* in $C(\phi)$).

Lemma 3.1. *Let S be a tomonoid. If $d \in D(\phi)$, then $\phi(d^-) <_S \phi(d^+)$.*

Proof. Suppose $d \in D(\phi)$. Then $d = y - x$, for some $x, y \in F_N(I)$ with $\phi(x) <_S \phi(y)$. Since $x, y \geq_\ell 0$, $d^- \leq x$. Thus, setting $e := x - d^-$, we have $e \in F_N(I)$. We have $x = d^- + e$ and $y = d^+ + e$, and therefore $\phi(d^- + e) <_S \phi(d^+ + e)$. This implies $\phi(d^-) + \phi(e) <_S \phi(d^+) + \phi(e)$. Since in a tomonoid the strict order relation is cancellative, $\phi(d^-) <_S \phi(d^+)$. ∎

Suppose S is a tomonoid, $\phi : F_N(I) \to S$ and $d \in D(\phi)$. Let $e \in F_N(I)$. It is possible that $\phi(d^- + e) = \phi(d^+ + e)$, since two different elements of S may have the same translate by $\phi(e)$. Thus, in general, if $y - x \in D(\phi)$, then we can only conclude that $\phi(x) \leq_S \phi(y)$. From $\phi(x) \leq_S \phi(y)$, no conclusion about the containment of $y - x$ in $D(\phi)$ is possible.

If ϕ is surjective, then the full tomonoid structure of S is determined by D and ϕ. To be more precise:

Lemma 3.2. *Let S be a monoid, let $\leq_1$ and $\leq_2$ be two TITO's on S, and assume $\phi : F_N(I) \to S$ is surjective. The identity map on S is an order-isomorphism from $(S, \leq_1)$ to $(S, \leq_2)$ if and only if $D(\phi, <_1) = D(\phi, <_2)$.*

Proof. Every element of S is $\phi(z)$ for some $z \in F_N(I)$. If $\phi(x) \leq_1 \phi(y)$ then either $y - x \in D(\phi, <_1) = D(\phi, <_2)$ or $\phi(x) = \phi(y)$. In either case, $\phi(x) \leq_2 \phi(y)$. Similarly, $\phi(x) \leq_2 \phi(y)$ implies $\phi(x) \leq_1 \phi(y)$. ∎

Lemma 3.2 does not generalize to TIPO's, since the cancellative property of the strict order relation is needed.

In general, $D(\phi)$—unlike $C(\phi)$—is *not* closed under addition. If $d_1, d_2 \in D(\phi)$, then $d_1 + d_2$ may or may not be in $D(\phi)$. (This makes TITO's on monoids far more complex than TITO's on abelian groups.) Nonetheless, we have the following:

Lemma 3.3. *Let S be a tomonoid, and suppose $d_1, d_2 \in D(\phi)$. If ${d_1}^+ + d_2 \geq_\ell 0$ then $d_1 + d_2 \in D(\phi)$.*

Proof. If $d_i \in D$, then $\phi({d_i}^-) <_S \phi({d_i}^+)$. If ${d_1}^+ + d_2 \geq_\ell 0$, then ${d_1}^+ - {d_2}^- \in F_N(I)$, so $\phi({d_1}^+ - {d_2}^-)$ is defined. Then

$$\phi({d_1}^+) = \phi({d_2}^-) + \phi({d_1}^+ - {d_2}^-) \leq_S \phi({d_2}^+) + \phi({d_1}^+ - {d_2}^-) = \phi({d_1}^+ + d_2),$$

and so

$$\phi({d_1}^-) <_S \phi({d_1}^+ + d_2).$$

Thus,

$${d_1}^+ + d_2 - {d_1}^- = d_1 + d_2 \in D(\phi).$$

∎

A good way to understand this lemma is to think in terms of addition of directed segments. For $a, b \in \mathbf{N}^n$, let $(a \to b)$ denote the directed segment from a to b. We add segments as we add geometric vectors—by translation and concatenation:

$$(a \to b) + (c \to d) = (a \to b) + (b \to (d + b - c)) = (a \to (b + d - c)).$$

For $d \in D(\phi)$, we think of d as represented by $(d^- \to d^+)$. The addition rule reads: $(d_1^- \to d_1^+) + (d_2^- \to d_2^+) = (d_1^- \to (d_1^+ + d_2))$. Thus, $d_1 + d_2$ belongs to $D(\phi)$ if the concatenation of $(d_1^- \to d_1^+)$ and the appropriate translate of $(d_2^- \to d_2^+)$ terminates at a point in $\mathbf{N}^n$.

4. Formally integral tomonoids.

Suppose that S is a monoid and $\leq_S$ is a TITO on S. We say $\leq_S$ is *formally integral* (or that $(S, \leq_S)$ is a formally integral tomonoid) if there is a set I, a monoid surjection $\phi : F_N(I) \twoheadrightarrow S$ and a TITO $\leq_0$ on $F_N(I)$ such that $x \leq_0 y$ implies $\phi(x) \leq_S \phi(y)$. Obviously, for any set J, any TITO on $F_N(J)$ is formally integral. Also obvious is the fact that a quotient of a formally integral tomonoid is formally integral. (Reminder: if S and T are tomonoids, we say that T is a quotient of S if there is a surjective tomonoid morphism from S onto T. This is not the meaning of "homomorphic image" used in model theory (as in [CK], page 70). The latter allows, for example, that if $(S, \leq_S)$ is a tomonoid and R is any relation whatsoever such that $\leq_S \subseteq R$ (in $S \times S$), then (S, R) is a "homomorphic image" of $(S, \leq_S)$.)

Proposition 4.1. *If $(S, \leq_S)$ is formally integral and $\psi : F_N(J) \to S$ is any homomorphism, there is a TITO on $F_N(J)$ such that ψ preserves order.*

Proof. We may find a set I, a monoid surjection $\phi : F_N(I) \twoheadrightarrow S$ and a TITO $\leq_0$ on $F_N(I)$ such that ϕ preserves order. Let $\psi : F_N(J) \to S$ be given. There is a monoid homomorphism $f : F_N(J) \to F_N(I)$ such that $\phi \circ f = \psi$. This has a unique extension to a homomorphism of **Q**-vector spaces $\hat{f} : F_Q(J) \to F_Q(I)$. Let $P \subseteq F_Q(I)$ be the positive cone of the unique extension of $\leq_0$ to $F_Q(I)$. Then $\hat{f}^{-1}(P \setminus \{0\})$ is contained in a pointed cone in $F_Q(J)$. By adding to $\hat{f}^{-1}(P \setminus \{0\})$ any T-cone in the kernel of $\hat{f}$, we obtain a T-cone in $F_Q(J)$. This T-cone determines a TITO on $F_Q(J)$ with respect to which $\hat{f}$ is an tomonoid morphism. With respect to the associated TITO on $F_N(J)$, then, $\psi = \phi \circ f$ is an tomonoid morphism. ▪

Corollary 4.2. *Any sub-tomonoid of a formally integral tomonoid is formally integral.*

Proof. Suppose U is a sub-tomonoid of S, and S is formally integral. Let I be a set of generators for U and let J be a set of generators for S containing I. There is a TITO $\leq_0$ on $F_N(J)$ such that $\phi : F_N(J) \twoheadrightarrow S$ is order preserving. The restrictions of $\leq_0$ and ϕ to $F_N(I)$ give the data needed to show that U is formally integral. ▪

The following lemma is used to prove the next proposition, but it is interesting in its own right.

Lemma 4.3. *The TITO's on $F_N(I)$ with respect to which $\phi : F_N(I) \to S$ is order-preserving are in one-to-one correspondence with the T-cones in $F_Q(I)$ that contain $D(\phi)$.*

Proof. Suppose $\leq_0$ is a TITO on $F_N(I)$. Then ϕ is order-preserving if and only if:

$$\text{for all } x, y \in F_N(I),\ x \leq_0 y \text{ implies } \phi(x) \leq_S \phi(y).$$

The contrapositive of this is equivalent to

$$(\phi \times \phi)^{-1}(<_S) \subseteq <_0$$

(here, treating an order $\leq$ on any set X as a subset of $X \times X$). And finally, this is the case if and only if $D(\phi) \subseteq P(\leq_0)$. ▪

Our next proposition provides useful criteria for formal integrality.

Proposition 4.4. *For any tomonoid S, the following are equivalent:*

i) *S is formally integral;*

ii) *for some surjective $\phi : F_N(I) \to S$, $C(\phi) \subseteq F_Q(I)$ is pointed;*

iii) *for all $\phi : F_N(I) \to S$, $C(\phi) \subseteq F_Q(I)$ is pointed.*

Proof. This is immediate from Proposition 4.1 and Lemma 4.3. ▪

The following answers the question that appears on line 24 of table 1.1 of [G].

Corollary 4.5. *Any totally ordered abelian group is formally integral. Any cancellative tomonoid is formally integral.*

Proof. Every cancellative tomonoid is a sub-tomonoid of a totally ordered abelian group (*e.g.*, [Sa], p.4), so by 4.2 it is enough to prove the first assertion. Let S be a totally ordered abelian group and let $\phi : F_N(I) \to S$ be a surjective homomorphism. We need to show that $D(\phi)$ is contained in a pointed cone. Let $P^\bullet := \{ s \in S \mid s > 0 \}$ and let $\hat{\phi} : F_Z(I) \to S$ be the homomorphism induced by ϕ. Then, $D(\phi) = \hat{\phi}^{-1}(P^\bullet)$. Suppose $d_i \in D(\phi)$, $\lambda_i \in \mathbf{N}$ and $\sum_i \lambda_i d_i = 0$. Then $\sum_i \lambda_i \hat{\phi}(d_i) = 0$ and $\hat{\phi}(d_i) \in P^\bullet$, so all the λ_i vanish. ▪

The following proposition is a special case of the compactness theorem of first order logic. The theory we have developed, however, enables us to give a simple proof with no reference to logicians' concepts.

Proposition 4.6. *Every positive tomonoid that is not formally integral contains a finitely generated sub-tomonoid that is not formally integral.*

Proof. Suppose that S is not formally integral and $\phi : F_N(I) \to S$ is a surjection. Since $C(\phi, <_S)$ is not pointed, there is a relation of the form $0 = a_1 d_1 + \cdots a_k d_k$ with $d_i \in D(\phi, <_S)$ and $a_i \in \mathbf{N}$. The d_i are determined by finitely many elements of S, and therefore the offending relation exists in the portion of S generated by a finite set $\{ \phi(\varepsilon_j) \mid j \in J \}$. ▪

The following shows that that every positive tomonoid that is not formally integral has associated with it a tomonoid whose failure to be formally integral is due solely to order relations involving its largest finite element.

Proposition 4.7. *Let S be a positive tomonoid.*

a) *If S/v is formally integral and $u \leq_S v$ then S/u is also formally integral.*

b) *Suppose that $[0, v)$ does not have a largest element. If S/u is formally integral for all $u \in [0, v)$, then S/v is formally integral.*

c) *If S is a finitely generated positive tomonoid that is not formally integral, then there is an element $u_0 \in S$ having an immediate successor v_0 such that S/u_0 is formally integral and S/v_0 is not formally integral.*

Proof. **a)** The first assertion is immediate from the existence of the tomonoid morphism $\pi_u : S/v \to S/u$. **b)** Suppose that S/u is formally integral for all $u \in [0, v)$. Let $\phi : F_N(I) \twoheadrightarrow S/v$ be a surjection, let $D := D(\phi)$ and let C denote the cone in $F_Q(I)$ generated by D. For each $u \in [0, v)$, the difference set $D_u := D(\pi_u \circ \phi)$ generates a pointed cone—call it C_u—in $F_Q(I)$. Moreover, if $u_0 \leq_S u_1$, then $D_{u_0} \subseteq D_{u_1}$, so $C_{u_0} \subseteq C_{u_1}$. Now, if $[0, v)$ does not have a largest element then D is equal to the union of the chain $\{ D_u \mid u \in [0, v) \}$, and C is equal to the union of the chain $\{ C_u \mid u \in [0, v) \}$. But an increasing union of pointed cones is a pointed cone, so D is contained in a pointed cone. **c)** Let v_0 be the least element in $\{ v \in S \mid S/v \text{ is not formally integral} \}$. By part 2), $[0, v_0)$ has a largest element u_0. ▪

Suppose S is a nontrivial tomonoid generated by a single element e. Replacing the order with its opposite if necessary, we may assume $0_S <_S e$. Then it is easy to show that either S is isomorphic to $\mathbf{N}$ (in the natural order) or S is isomorphic to $\mathbf{N}/a$ for some integer a. From this, it is easy to see that S is formally integral. Later, we give examples of three-generator positive tomonoids that are not formally integral. As for the remaining possibility, we have the following.

Theorem 4.8. *Every tomonoid generated by two elements is formally integral.*

Proof. Suppose $\phi : \mathbf{N}^2 \to S$ is a surjective monoid homomorphism. Unless S is trivial, after exchanging coordinates in $\mathbf{N}^2$ and/or replacing $\leq_S$ with its opposite if necessary, we may assume $0_S <_S \phi((1,0))$. If $\phi((0,1)) = 0_S$, then S is generated by a single element. Assuming that neither generator is mapped to $)_S$, there are two cases to consider: either S is positive or it is not. We consider these in turn. (It is illuminating to rephrase the following in terms of addition of directed segments, as in remarks following lemma 3.3. This leads to an appealing, intuitive geometric argument. We have chosen a more formal presentation for the sake of rigor.)

Case 1: $0_S <_S \phi((0,1))$. In this case, S is positive, and therefore $D(\phi)$ can contain no pair (i, j) with both $i \leq 0$ and $j \leq 0$. If S is not formally integral, $D(\phi)$ is *not* contained in a pointed cone. Then there are $d = (-d_1, d_2)$ and $e = (e_1, -e_2)$ such that:

a) $d, e \in D(\phi)$,
b) $d_1, d_2, e_1, e_2 \in \mathbf{N}$, and
c) either:
  - $md = -ne$ for some positive integers m and n, or
  - the angle formed by d and e (with vertex at $(0,0)$) has $(-1,-1)$ in its interior.

In either of the alternatives allowed by the last condition, we have

c′) $d_1 e_2 \geq d_2 e_1 \geq 0$,

so one of the following pairs of inequalities must be satisfied:

i) $e_1 < d_1$ and $e_2 < d_2$, or
ii) $e_1 \leq d_1$ and $e_2 \geq d_2$, or
iii) $e_1 > d_1$ and $e_2 > d_2$.

Note that *ii*) means that the directed segments $(d^- \to d^+)$ and $(e^- \to e^+)$ cross one another. We will show that this is impossible. In the other two cases, one of the directed segments is long enough that they do not cross. We will show that successive additions produce a pair that does cross. In detail, if *i*) holds, then $d + e$ is in $D(\phi)$ (by lemma 3.3), and $d + e$ and e satisfy the original hypotheses on d and e. Moreover, $d + e$ is shorter than d. Similarly, if *iii*) holds, we may replace d and e with d and $d + e$, and $d + e$ is shorter than e. But such replacements cannot be repeated indefinitely, so after a finite number of replacements, case *ii*) must arise. However, *ii*) is impossible because it implies $\phi(e^-) \leq_S \phi(e^+) \leq_S \phi(d^-) \leq_S \phi(d^+) \leq_S \phi(e^-)$, forcing all to be equal, which by assumption they are not. (Note that we use the hypothesis that S is positive for the inequalities $\phi(e^+) \leq_S \phi(d^-)$ and $\phi(d^+) \leq_S \phi(e^-)$.)

Case 2: $\phi((0,1)) <_S 0_S$. In this case, $D(\phi)$ contains all integer pairs (i,j) with $0 \leq i$ and $j \leq 0$ and contains no pair (i,j) with both $i \leq 0$ and $0 \leq j$. If S is not formally integral, $D(\phi)$ is *not* contained in a pointed cone, so there are $d = (d_1, d_2)$ and $e = (-e_1, -e_2)$ in $D(\phi)$, such that:

- $d_1 \neq 0$ and $e_2 \neq 0$,
- d_1, d_2, e_1, e_2 are all greater than or equal to 0, and
- either:
    - $md = -ne$ for some integers m and n, or
    - the angle formed by d and e (with vertex at $(0,0)$) has $(-1,1)$ in its interior.

Since $e^+ = 0$, $d + e \in D(\phi)$, by Lemma 3.3. There are two possibilities: either $d + e \in \{ (i,j) \mid i \leq 0 \;\&\; 0 \leq j \}$—which is already the contradiction we seek—or one of the pairs, $\{ d, d{+}e \}$ or $\{ d{+}e, e \}$, satisfy the original hypotheses on the pair $\{ d, e \}$, but with shorter vectors. As in case 1, we may repeat the process until a contradiction is reached. ∎

5. Examples on 3 generators.

In this section, we give examples of three-generator monoids that are not formally integral. Previously, the minimum number of generators in any known example was 4. Here is a brief history. [HI] contains an amazing example of a tomonoid with 9 generators and 80 elements that is not formally integral but which has no 8-generator sub-tomonoid that fails to be formally integral. The paper [GIR] contains an example of a 4-generator tomonoid with 16 elements that is not formally integral. In [I1], Isbell generalizes the construction of [HI] to obtain, for each $n \geq 6$, a finite tomonoid with n generators that is not formally integral and has no sub-tomonoid on fewer than n generators that fails to be formally integral. (This means, incidentally, that the first order theory of formally integral tomonoids does not have a axiomatization by a set of positive sentences involving finitely many variables.) In [I2], Isbell constructs a 4-generator tomonoid different from the one in [GIR] that is not formally integral.

For any $a, b, c, d \in \mathbf{N}$, $\langle a, b, c \rangle$ will denote the sub-tomonoid of $\mathbf{N}$ generated by a, b and c, and $\langle a, b, c \rangle / d$ will denote the tomonoid obtained by identifying all elements of $\langle a, b, c \rangle$ that are greater than or equal to d with infinity (cf. the definition in section 1).

Let $\{32^*\} \cup \langle 9, 12, 16 \rangle / 30$ denote the monoid obtained from $\langle 9, 12, 16 \rangle / 30$ by adjoining one additional element, denoted 32^*. This element is to satisfy $16 + 16 = 32^*$ and the whole monoid is to be ordered as follows

$$0 < 9 < 12 < 16 < 18 < 21 < 24 < 25 < 27 < 28 < 32^* < \infty.$$

All the relations that do not involve 32^* are as in $\mathbf{N}$, so to check that this order is translation-invariant it suffices to note that any positive translate of the pair $(32^*, \infty)$ is (∞, ∞) and that if $(32^*, \infty)$ is a positive translate of (x, y), then $x < y$. Now, in this tomonoid we have the following relations:

$$24 = 2 \cdot 12 \;<\; 9 + 16 = 25$$
$$27 = 3 \cdot 9 \;<\; 12 + 16 = 28$$
$$32^* = 2 \cdot 16 \;<\; 2 \cdot 9 + 12 = \infty.$$

But note that in $\mathbf{N}$, the following pair of inequalities

$$\begin{aligned} 2\,y &\le x + z \\ 3\,x &\le y + z \end{aligned}$$

implies that

$$3\,x + 2\,y \le x + y + 2\,z,$$

and hence that

$$2\,x + y \le 2\,z.$$

Therefore, $\{32^*\} \cup \langle\, 9, 12, 16\,\rangle/30$ cannot be a quotient of any tomonoid with underlying monoid $\mathbf{N}^3$. In fact, if $\phi : \mathbf{N}^3 \to \{32^*\} \cup \langle\, 9, 12, 16\,\rangle/30$ is given by $\phi((i, j, k)) = 9\,i + 12\,j + 16\,k$, then the cone $C(\phi) \subseteq \mathbf{Q}^3$ generated by $D(\phi)$ is the closed half-space $\{\,(x, y, z) \mid 0 \le 3\,x + 4\,y + 5\,z\,\}$.

This example has 12 elements. The following quotient of it provides an example of a non-formally-integral tomonoid with only 9 elements:

$$0 < 9 < 12 < 16 < 18 < \{21, 24\} < \{25, 27\} < \{28, 32^*\} < \infty.$$

As mentioned in the introduction, Whipple has shown that all positive tomonoids with 8 or fewer elements are formally integral.

Our second example is a three-generator monoid for which the cone generated by the difference set is all of $\mathbf{Q}^3$. It is constructed from $\langle\, 20, 25, 27\,\rangle$. Here is table of the first 22 elements in this monoid, together with their factorizations ($s = 20\,i + 25\,j + 27\,k$):

$\sharp$		s		i	j	k
0		0		0	0	0
1		20		1	0	0
2		25		0	1	0
3		27		0	0	1
4		40		2	0	0
5		45		1	1	0
6		47		1	0	1
7		50		0	2	0
8		52		0	1	1
9		54		0	0	2
10		60		3	0	0
11		65		2	1	0
12		67		2	0	1
13		70		1	2	0
14		72		1	1	1
15		74		1	0	2
16		75		0	3	0
17		77		0	2	1
18		79		0	1	2
19		80		4	0	0
20		81		0	0	3
21		85		3	1	0
22		87		3	0	1

The monoid $\{85^*\} \cup \langle\, 20, 25, 27 \,\rangle / 81$ is constructed as in the preceding example, with $85^* = 3 \cdot 20 + 25$ the largest element that is less than $\infty = 81$. It is easy to check that this is a tomonoid. We have the relations:

$$\begin{aligned} 47 = 20 + 27 \;&<\; 2 \cdot 25 = 50 \\ 79 = 25 + 2 \cdot 27 \;&<\; 4 \cdot 20 = 80 \\ 85^* = 3 \cdot 20 + 25 \;&<\; 3 \cdot 27 = \infty. \end{aligned}$$

But in $\mathbf{N}$, the following pair of inequalities

$$\begin{aligned} x + z &\le 2\,y \\ y + 2\,z &\le 4\,x \end{aligned}$$

implies that

$$x + y + 3\,z \le 4\,x + 2\,y,$$

and hence that

$$3\,z \le 3\,x + y.$$

Therefore, $\{85^*\} \cup \langle\, 20, 25, 27\,\rangle/81$ cannot be a quotient of any tomonoid with underlying monoid $\mathbf{N}^3$. In fact, if $\phi : \mathbf{N}^3 \to \{85^*\} \cup \langle\, 20, 25, 27\,\rangle/81$ is given by $\phi((i,j,k)) = 20\,i + 25\,j + 27\,k$, then $D(\phi)$ contains $(4,-1,-2)$, $(-1,3,-2)$, $(-4,0,3)$, as well as $(-3,-1,3)$—the last as a consequence of the relation $85^* < 81 = \infty$. The first three vectors are a basis for $\mathbf{Z}^3$, and twice the first plus the second and third is $(3,1,-3)$, so $(3,1,-3)$ is in the interior of the cone generated by the first three. Thus, the cone generated by $D(\phi)$ is all of $\mathbf{Q}^3$.

6. Further criteria for formal integrality.

Suppose that S is a positive tomonoid and $S_1, S_2 \subseteq S$ are sub-tomonoids such that $S = \langle\, S_1, S_2\,\rangle$, *i.e.*, S is generated by $S_1 \cup S_2$. Suppose further that S_1 and S_2 are formally integral. Under what conditions can we conclude that S formally integral? Theorem 6.1, which is a generalization of 4.8, addresses an instance of this problem that is particularly relevant to intended applications to ring theory, as we explain in §8. The proof follows the pattern of the proof of 4.8, but with substantial added complications.

Theorem 6.1. *Suppose that $S = \langle\, S_1, c\,\rangle$, where S_1 is (isomorphic to) a sub-tomonoid of $\mathbf{N}/w$ and c is an element of S. Let $\pi : \mathbf{N} \to \mathbf{N}/w$ be the canonical map, and let $T := \pi^{-1}(S_1)$. Suppose that there is a positive integer $M \in T$ such that for any integer n, $M \le n$ implies $n \in T$ (hence, $\pi(n) \in S_1$ for all integers $n \ge M$). Suppose also that $\pi(M) <_S c$. Then S is formally integral.*

Proof: Let $D \subseteq \mathbf{Z} \oplus \mathbf{Z}$ be the set consisting of all pairs that are of one of the following forms:

- $(-t, n)$, where $t \in T$, $n \in \mathbf{N}$ and $\pi(t) <_S n\,c$,
- $(t, -n)$, where $t \in T$, $n \in \mathbf{N}$ and $n\,c <_S \pi(t)$.

The significance of this is that the inequalities that define these pairs, together with the TITO's on S_1 and $\langle c\rangle$, generate $\le_S$. and more conceptual proof can will give

We complete the proof by proving two assertions. In the following, three TITO's are mentioned: $<$ and $\le$ (no subscripts) refer to the natural order in $\mathbf{N}$ or $T \subseteq \mathbf{N}$, $<_S$ and $\le_S$ refer to the given TITO on S, and finally $<_1$ and $\le_1$ appearing in assertion 2 refer to a TITO on $T \oplus \mathbf{N}$. The homomorphism

$$T \oplus \mathbf{N} \to S; (t, n) \mapsto \pi(t) + n\,c$$

is used frequently.

Assertion 1: D is contained in a P-cone. We prove this by an argument that is similar to that used in the case 1 part of the proof of 4.8. Suppose that assertion 1 fails. Since $D \cup \{0, 0\}$ contains $T \oplus \{0_{\mathbf{N}}\}$ and $\{0_T\} \oplus \mathbf{N}$, the cone in $\mathbf{Q} \oplus \mathbf{Q}$ that D generates must contain $T \oplus \mathbf{N}$. Therefore, there must be $d = (-d_1, d_2)$ and $e = (e_1, -e_2)$ that satisfy condition c) from the proof of 4.8. As argued there, one of the three pairs of inequalities i), ii) or iii) must be satisfied. Suppose that i) holds. Then $e_2 \ne 0$, for if $e_2 = 0$ then $e_1 \ne 0$ so $d_2 = 0$ by c') (in proof of 4.8), but this contradicts i). Using c') again and i)

$$(e_2 + 1)\,e_1 \le e_2\,d_1$$

so

$$e_1 + \frac{e_1}{e_2} \leq d_1.$$

From

$$e_2\, c <_S \pi(e_1),$$

it follows that

$$e_2 \pi(M) <_S \pi(e_1)$$

so

$$e_2\, M < e_1.$$

Therefore, $e_1 + M \leq d_1$, so $d_1 - e_1 \in T$. Then the same argument as in the proof of lemma 3.3 shows that $d + e \in D$, and the pair $(d + e, e)$ satisfies the original conditions on (d, e). Suppose that *iii*) holds. It is not possible that $e_1 - d_1 \leq M$, because if so then

$$(d_2 + 1)\, c \leq_S e_2\, c <_S \pi(e_1) \leq_S \pi(d_1 + M),$$

while on the other hand

$$\pi(d_1) <_S d_2\, c,$$

and

$$\pi(M) \leq_S c$$

so

$$\pi(d_1 + M) \leq_S (d_2 + 1)\, c.$$

Hence $e_1 - d_1 > M$, and therefore $e_1 - d_1 \in T$. Using the proof of 3.3, $d + e \in D$ and the pair $(d, d + e)$ satisfies the original conditions on (d, e). As in the proof of 4.8, the outcomes of the two cases considered show that eventually case *ii*) must arise, but as in 4.8, this is impossible. This completes the proof of assertion 1.

Assertion 2: The homomorphism

$$T \oplus \mathbf{N} \to S; (t, n) \mapsto \pi(t) + n\, c$$

is order-preserving for any TITO $\leq_1$ *on* $T \oplus \mathbf{N}$ *that is induced by a T-cone in* $\mathbf{Z} \oplus \mathbf{Z}$ *that contains* D. To prove this, it suffices to show that for all $s, t \in T$, $m, n \in \mathbf{N}$:

$$(*)\ \pi(s) + m\, c <_S \pi(t) + n\, c \quad \textit{implies} \quad (**)\ (s, m) <_1 (t, n).$$

Let $k := \min\{m, n\}$. From $(*)$, it follows that $\pi(s) + (m - k)\, c <_S \pi(t) + (n - k)\, c$. If we succeed in deducing from this that $(s, m - k) <_1 (t, n - k)$, then we can conclude $(**)$ (from translation-invariance and the fact that $(s, m) \neq (t, n)$). In fact, if $m = n$, then $(**)$ follows immediately (from these). Otherwise, we need to consider two cases: *Case 1:* $\pi(s) + m\, c <_S \pi(t)$ *and* $m \neq 0$. Then $\pi(s + m\, M) <_S \pi(t)$, so there is $u \in T$ such that $u + s = t$. By cancellation in S, $m\, c <_S \pi(u)$, so $(u, -m) \in D$. Thus, $(0, m) <_1 (u, 0)$, so $(s, m) <_1 (t, 0)$. *Case 2:* $\pi(s) <_S \pi(t) + n\, c$ *and* $n \neq 0$. If $\pi(s) \leq_S \pi(t)$, then $(s, 0) \leq_1 (t, 0) \leq_1 (t, n)$ and since $(s, 0) \neq (t, n)$, we have $(s, 0) <_1 (t, n)$. If $\pi(t) <_S \pi(s)$, we consider sub-cases. If $s < t + n\,M$, then $(s, 0) <_1 (t + n\,M, 0) <_1 (t, n)$, which is all that is needed. If $s \geq t + n\,M$, there is $u \in T$ such that $s = t + u$. By cancellation in S, $\pi(u) <_S n\,c$, so $(-u, n) \in D$. Thus $(u, 0) <_1 (0, n)$, so $(s, 0) <_1 (t, n)$. This completes the proof of assertion 2.

To finish the proof of the theorem, it suffices to observe that $(T \oplus \mathbf{N}, \leq_1)$ is a sub-tomonoid of a formally integral tomonoid and that S is a quotient of $(T \oplus \mathbf{N}, \leq_1)$. ▪

7. Nil tomonoids.

In this section, we examine the structure of finite nil tomonoids and explain and justify a conjecture that states that any obstruction to formal integrality that can occur in a positive tomonoid occurs in a nil tomonoid. Let S be a positive tomonoid and let $s \in S$.

If S is archimedean, it may or may not have a largest element. A *nil* tomonoid is a positive archimedean tomonoid that has a largest element. Thus, a tomonoid S is nil if and only if it is positive, has an absorbing element ∞ and for all $x \in S \setminus \{0\}$ there is $n \in \mathbf{N}$ such that $nx = \infty$. A nil tomonoid is finitely generated if and only if it is finite.

In the remainder of this section, S will be a finite nil tomonoid with minimal generating set $\{s_1 <_S \cdots <_S s_n\}$ and $\phi : \mathbf{N}^n \twoheadrightarrow S$ will be the surjection defined by setting $\phi(\varepsilon_i) = s_i$. As before, $C(\phi)$ denotes the cone in $\mathbf{Q}^n$ generated by $D(\phi)$.

Lemma 7.1. *$C(\phi)$ is a polyhedral convex cone, which is pointed if and only if S is formally integral.*

Proof. The set $D(\phi)$ is not finite, since there are infinitely many distinct differences $y - x$ with $\phi(x) < \phi(y) = \infty$. Our task is to show that $C(\phi)$, nonetheless, has only finitely many extreme rays. This will follow, as we now show in detail, from the fact that lattice ideals in $\mathbf{N}^n$ are finitely generated. Let $E := \phi^{-1}(\infty)$. Then E is an ideal in $\mathbf{N}^n$ with finite complement $E' \subseteq \mathbf{N}^n$. There is a unique minimal generating set G for E, and G is finite. (In other words, $E = \cup\{\, g + \mathbf{N}^n \mid g \in G\,\}$, and equality fails if any element of G is omitted.) Let $x \in E'$, and let $y \in E$. Any cone that contains all the ε_i as well as $\{\, g - x \mid g \in G\,\}$ also contains $y - x$. Thus, $C(\phi)$ is generated by differences $y - x$ that satisfy $x \in E'$ and $y \in E' \cup G$. There are only finitely many such differences. The last assertion is immediate from 4.4. ▪

If instead of ϕ we had started with an arbitrary surjection, then E' could fail to be finite, and the conclusion of the lemma could fail. It suffices to require that $\phi^{-1}(0_S) = \{0_{\mathbf{N}^m}\}$; we leave the details to the reader.

In the next two results, we use the dual cone $C^*(\phi)$ of $C(\phi)$. The definition is

$$C^*(\phi) := \{\, \xi \in \mathbf{Q}^n \mid \ \xi \cdot c \geq 0 \text{ for all } c \in C(\phi)\,\}.$$

Observe that $C^*(\phi)$ *is contained in* the positive orthant of $\mathbf{Q}^n$, since—S being positive—$C(\phi)$ *contains* the positive orthant of $\mathbf{Q}^n$. The *interior* of $C^*(\phi)$ is the set of ξ such that $\xi \cdot c > 0$ for all $c \in C(\phi)$.

Proposition 7.2. *Every finite formally integral nil tomonoid is a quotient of a sub-tomonoid of* $\mathbf{N}$.

Proof. Assume S is formally integral. Using the notation from the previous proof, let $a = (a_1, \dots a_n) \in \mathbf{N}^n$ be any integral vector in the interior of the dual cone of $C(\phi)$. Now, $\phi(x) <_S \phi(y)$ implies $a \cdot x < a \cdot y$ or, taking the contrapositive,

$$a \cdot y \le a \cdot x \text{ implies } \phi(y) \le_S \phi(x). \qquad (*)$$

Let T_a be the sub-tomonoid of $\mathbf{N}$ generated by the components a_i of a. Define $\psi_a : T_a \to S$ by $\psi_a(a \cdot x) := \phi(x)$; ψ_a is well-defined, since if $a \cdot x = a \cdot x'$, then $\phi(x) = \phi(x')$, by $(*)$. Since $\psi_a(a \cdot x + a \cdot y) = \psi_a(a \cdot (x + y)) = \phi(x + y) = \phi(x) + \phi(y) = \psi_a(a \cdot x) + \psi_a(a \cdot y)$, ψ_a is additive, and $(*)$ ensures that ψ_a is order-preserving. ∎

Let $\mathcal{T}$ denote the set of pairs (T, ψ), where $T = \langle a_1, \dots, a_n \rangle \subseteq \mathbf{N}$ and $\psi : T \to S$ is a surjective tomonoid morphism with $\psi(a_i) = s_i$. For each integral element $a = (a_1, \dots, a_n)$ in the interior of $C^*(\phi)$, let (T_a, ψ_a) be the pair described in the preceding proof.

Corollary 7.3. *Suppose S is a formally integral finite nil tomonoid. The function*

$$a \mapsto (T_a, \psi_a)$$

from the set of integer points in the interior of $C^*(\phi)$ to $\mathcal{T}$ is a bijection.

Proof. To see that it is injective, observe that $a_1 < \cdots < a_n$ is a minimal generating set for T_a. Suppose $a \neq a'$. Without loss of generality, $a_i = a'_i$ for $i < m$, and $a_m < a'_m$. Then $a_m \notin T_{a'}$, so $T_a \neq T_{a'}$. To see that it is surjective, suppose that $(\langle b_1 < \cdots < b_n \rangle, \psi_b) \in \mathcal{T}$, so $\psi_b(b_i) = s_i$. For any $d \in D(\phi)$, $d^- \cdot s <_S d^+ \cdot s$, so $\psi(d^- \cdot b) <_S \psi(d^+ \cdot b)$, so $0 < d \cdot b$. This means that b is in the interior of $C^*(\phi)$. Therefore, $a \mapsto (T_a, \psi_a)$ is surjective. ∎

Let us consider the meaning of 7.3 from a geometric standpoint.

The corollary makes it easy to describe the sub-tomonoids of $\mathbf{N}$ of which a given nil tomonoid is a quotient. We illustrate with an example. Let $S = \langle 9, 12, 16 \rangle / 33 = \{0 < 9 < 12 < 16 < 18 < 21 < 24 < 25 < 27 < 28 < 30 < 32 < 33 = \infty\}$, and let $\phi : \mathbf{N}^3 \to S$ be defined by $\phi(\varepsilon_1) = 9$, $\phi(\varepsilon_2) = 12$ and $\phi(\varepsilon_3) = 16$. Then $C(\phi)$ has edges $(1, -2, 1)$, $(1, 2, -2)$ and $(-3, 1, 1)$, and $C^*(\phi)$ has edges $(2, 3, 4)$, $(4, 5, 7)$ and $(3, 4, 5)$. The shortest integral vector in the interior of $C^*(\phi)$ is $(9, 12, 16)$. Since $C^*(\phi)$ is unimodular, the other interior integer points can be written in the form $(a_1, a_2, a_3) = \beta_1(2, 3, 4) + \beta_2(4, 5, 7) + \beta_3(3, 4, 5)$, where $\beta_1, \beta_2, \beta_3 \in \mathbf{N} \setminus \{0\}$. For any such point, $\langle a_1, a_2, a_3 \rangle / (a_1 + 2a_2)$ is order-isomorphic to S.

If, as in this example, every finite element of S has "unique factorization" (*i.e.*, for all $b, b' \in \mathbf{N}^n$, $\sum_{i=1}^n b_i s_i = \sum_{i=1}^n b'_i s_i <_S \infty$ implies $b = b'$), then ψ factors as $T_a \to T_a/d_a \to S$, where d_a is the least element of T_a that ψ_a takes to ∞, and $T_a/d_a \to S$ will be a bijective tomonoid morphism—in other words, S will be order-isomorphic to T_a/d_a. Even if S does not have unique factorization, we may find an integral vector b in the interior of $C^*(\phi)$ so that T_b/d_b does. Simply choose any a in the interior of $C^*(\phi)$. If $\delta \in \mathbf{Q}^n$ has sufficiently small components then $a + \delta$ remains in the interior of $C^*(\phi)$. Moreover, for any finite set $X \subseteq \mathbf{N}^n$ we can choose arbitrarily small δ so that the numbers $\xi \cdot (a + \delta)$ ($\xi \in X$) are all distinct. Let X be the set of factorizations of all the finite elements of S and let $b = k(a + \delta)$ where k is the least common multiple of the denominators in $a + \delta$.

We now join the results of this section with the previous section. Suppose S is a finite nil tomonoid that is not formally integral. There are elements $u <_S v$ of S such that v is the immediate successor of u, and S/u is formally integral and S/v is not. By the above, S/u is a quotient of a sub-tomonoid of $\mathbf{N}$. Let $s_1, \ldots, s_n$ be a minimal set of generators for S/u, and let $\langle a_1, \ldots, a_n \rangle \subseteq \mathbf{N}$ map onto S/u by $a_i \mapsto s_i$. Let d be the least element of $\langle a_1, \ldots, a_n \rangle$ that maps to $\infty_{S/u}$, and let $T := \langle a_1, \ldots, a_n \rangle / d$. We have a surjection $\pi : T \twoheadrightarrow S/u$ with $\pi^{-1}(\infty_{S/u}) = \{\infty_T\}$.

This section is not finished.

8. Applications to ordered rings.

After giving some definitions and historical background, we shall describe how the material on monoids presented in this paper relates to ordered rings. All rings discussed in this paper are commutative and to have a unit element. A totally ordered ring—or *toring*, for short—is a ring A equipped with a total order $\leq$ that satisfies the following conditions:

- for all $a, b, c \in A$, if $a \leq b$ then $a + c \leq b + c$, and
- for all $a, b, c \in A$, if $a \leq b$ and $0 \leq c$ then $ac \leq bc$.

A *toring morphism* is an order-preserving ring homomorphism between torings. An ideal I in a toring A is said to be *convex* if for all $x, y \in A$,

$$(0 \leq x \leq y \ \&\ y \in I) \Rightarrow x \in I.$$

The kernel of any toring morphism is convex and if I is convex, then A/I is naturally a toring.

In recent years, torings without nilpotents have played an important role in real algebraic geometry, but little consideration has be given to phenomena related to nilpotency in torings. We make a few historical observations connected to this. Totally ordered fields played a significant role in Hilbert's *Grundlagen der Geometrie*. This work, in fact, contained the original motivation for Hilbert's 17th problem; see [DP]. The modern algebraic theory of ordered fields was initiated by Artin and Schreier in their solution of this problem. Artin and Schreier called a field *formally real* if it admits a total order, or equivalently, if the negative of the unit element is not equal to a sum of squares. The concept of a formally real field has a generalization that is extremely useful in real algebraic geometry: a *real* ring is a ring that can be embedded in a product of formally real fields. The most important examples are the real coordinate rings of real algebraic varieties. Each total ordering of such a ring has a precise geometric meaning—it corresponds to a prime filter of closed semialgebraic sets; see [BCR]. This correspondence accounts for the importance of totally ordered domains in real algebraic geometry. It is obvious that a real ring has no nilpotent elements. With geometric applications in mind, Brumfiel derived a few basic results about orderings of rings with nilpotents in [B], but his observations in this connection do not appear to have fed into any further work in real algebraic geometry. Rings with nilpotents, however, do arise naturally in geometric settings. A ring of function germs (with respect to a prime filter of closed semialgebraic sets) modulo an ideal of germs vanishing to some (fixed) order is an example. (It is worth noting that any toring that is not a domain contains nilpotent elements, for if $0 \leq x \leq y$ and $xy = 0$, then $x^2 = 0$.)

In the purely algebraic theory of ordered rings, there are two important contributions prior to 1970 that are particularly relevant to torings with nilpotents. First is the work of Hion [Hi], and second is the work of Henriksen and Isbell [HI]. Together, these provide a strong connection between torings (possibly with nilpotents) and tomonoids.

We now summarize the relevant parts of Hion's work in modern notation. Let $(A, \leq)$ be a toring. Let A^+ denote the (multiplicative) tomonoid of elements of A that are greater than or equal to 0. Two elements $a, b \in A^+$ are said to be *additively* archimedean equivalent if there are $m, n \in \mathbf{N}$ such that $b \leq ma$ and $a \leq nb$. If this is so, we write $a \sim b$. It is easy to see that $\sim$ is an equivalence relation on A^+. The class of a is denoted $[a]$, and the set of all classes is denoted $\mathcal{H}(A)$. Obviously, $[a]$ is an interval in A^+, and one easily verifies that if $a \sim b$ and $c \sim d$, then $ac \sim bd$. Thus, $\sim$ is an order-congruence on A^+, so $\mathcal{H}(A)$ has a natural tomonoid structure. In order to have notation that is compatible with standard valuation theory (which this construction generalizes), in $\mathcal{H}(A)$ we use additive notation and the opposite of the natural order. Specifically, we define $[a] + [b] := [ab]$ and put $[a] \leq [b]$ iff $b \leq a$. We view $\mathcal{H}(A)$ as a tomonoid with this structure, and we call it the *Hion tomonoid of A*. Let $h : A \to \mathcal{H}(A)$ be defined by $h(a) := [|a|]$. As with the archimedean valuation on a totally ordered field, we have:

- $0 \leq a \leq b \quad \Rightarrow \quad h(b) \leq h(a) \leq h(0) = \infty$,
- $h(ab) = h(a) + h(b)$,
- $h(a + b) \geq \min\{\, h(a), h(b)\,\}$, with equality whenever $h(a) \neq h(b)$.

Let H be a tomonoid. We call H a *Hion tomonoid* if it has a largest element ∞, this element is absorbing and for all $x, y, z \in H$:

$$x + z = y + z \neq \infty \quad \Rightarrow \quad x = y.$$

A *Hion tomonoid morphism* is a tomonoid morphism $\phi : H \to K$ between Hion tomonoids that satisfies $\phi(x) = \phi(y) \neq \infty_J \;\; \Rightarrow \;\; x = y$ and $\phi(\infty_H) = \infty_J$. It follows from the isomporphism theorems for rings and the fact that the kernel of a toring morphism is convex that $\mathcal{H}$ is a functor from the category of torings to the category of Hion tomonoids. It is *not* full.

Theorem 8.1. *([Hi]) If A is a toring, then $\mathcal{H}(A)$ is a Hion tomonoid. Moreover, for any Hion tomonoid H, there is a toring A such that $\mathcal{H}(A)$ is isomorphic to H.*

Because Hion's original paper does not seem to have been translated from the Russian and therefore may not be accessible to some readers, we provide a sketch of the proof. The proof of the second part involves a construction with monoid rings, which we will explain now, before giving the the proof of 8.1. If R is a ring and S is a monoid, the monoid ring $R[S]$ is the set of all finite formal sums $r_1 X^{s_1} + \cdots + r_n X^{s_n}$, where X is an indeterminate, $r_i \in R$ and $s_i \in S$. Multiplication defined by the rule $X^s X^t = X^{s+t}$ and distributivity. Suppose S is a tomonoid. An element $g \in R[S]$ is said to be written in *normal form* when it is written as a sum of non-zero terms with exponents of ascending order:

$$g = r_1 X^{s_1} + \cdots + r_n X^{s_n}, \text{ with } r_i \neq 0 \text{ and } s_1 < s_2 < \cdots < s_n.$$

If S has a largest element ∞, let $R[S]_h$ denote the quotient of $R[S]$ obtained by identifying X^∞ with 0, ordered in such a way that an element $a_1X^{h_1} + \cdots$ in normal form is positive iff $a_1 > 0_A$.

Lemma 8.2. *If A is a toring without zero-divisors and H is a Hion tomonoid, then $A[H]_h$ is a toring. Moreover, $A[\ \ ]_h$ is a functor from the category of Hion tomonoids to torings.*

Proof. Sums of positive elements are clearly positive, so the only thing that requires checking is that products of positives are positive. Suppose $f = q_1X^{s_1} + \cdots + q_mX^{s_m}$ and $g = r_1X^{t_1} + \cdots + r_nX^{t_n}$ are positive. Then the product fg is a sum of terms of the form $q_ir_jX^{s_i+t_j}$. For any i and j, $s_1 + t_j$ and $s_i + t_1$ are between $s_1 + t_1$ and $s_i + t_j$, so if $s_1+t_1 = s_i+t_j$ then $s_1+t_1 = s_1+t_j = s_i+t_1$. Thus, by the Hion condition, if $s_1+t_1 \neq \infty$, then $s_1 + t_1 \neq s_i + t_j$ for any pair $(i, j) \neq (1, 1)$. With the assumption that A contains no zero-divisors, this shows that if $fg \neq 0$, then the leading term of fg is $q_1r_1X^{s_1+t_1}$. We leave the routine verification of the last assertion to the reader. ▪

Proof of 8.1. For the first part, suppose $a, b, c \in A$ and $h(a) < h(b)$. Then $n|b| \leq |a|$ for all $n \in \mathbf{N}$, and therefore $n|bc| \leq |ac|$ for all $n \in \mathbf{N}$. Suppose $h(bc) = h(ac)$. Pick $m \in \mathbf{N}$ such that $|ac| \leq m|bc|$. Then we get $n|bc| \leq m|bc|$ for all $n \in \mathbf{N}$, so $|bc| = 0$. For the second part, simply note that $\mathcal{H}(\mathbf{Z}[H]_h) = H$. ▪

We turn now to the contributions of Henriksen and Isbell. Motivated by Birkhoff's theorem on equational classes in universal algebra, Henriksen and Isbell show in [HI] that all totally ordered fields satisfy the same lattice-ring identities, and they introduce the class of *formally real f-rings*, which consists of those lattice-ordered rings that satisfy all lattice-ring identities that are true in a totally-ordered field.* For our purposes, the most important result of Henriksen-Isbell is the following—which may as well serve for us as a definition of formally real torings.

Theorem 8.3. *([HI]) A toring is formally real if and only if it is a quotient of a totally ordered domain by a convex ideal.* ▪

Henriksen-Isbell presented an example of a tomonoid algebra with 9 generators that is not formally real and that has the additional property that all eight-generator sub-torings are formally real. In a subsequent paper of remarkable originality, Isbell showed, by constructing algebras over appropriate (non-formally integral) finite Hion tomonoids, that the equational theory of formally real f-rings does not have a finite base, or even a base with a finite number of variables; see [I1]. The authors of [HI] and [I1] did not refer

* This paper is actually quite a *tour de force*. Among other things, they proved that the free formally real f-ring on n generators is isomorphic to the sub-f-ring of the ring of all real-valued functions on $\mathbf{R}^n$ generated by the polynomials with integer coefficients, thus solving a problem of Birkhoff and Pierce [BP]. While writing [HI], Henriksen and Isbell formulated the question now known as the Pierce-Birkhoff Conjecture; see [M] for historical notes. This conjecture is a sharpening of Henriksen-Isbell's description of the free formally real f-ring on n generators. It asserts that the free formally real $\mathbf{R}$-f-algebra on n generators is isomorphic to the $\mathbf{R}$-f-algebra of *all* piecewise polynomial functions on $\mathbf{R}^n$. At present, only the $n = 1$ and $n = 2$ cases are known to be true; see [Mah].

to the Hion condition, but it is implied by a "unique factorization" condition that they mention explicitly, which holds in tomonoids they use.

Definition. *An r-tomonoid is a tomonoid that is isomorphic (as a tomonoid) to some* S/K*, where* S *is a sub-tomonoid of a totally ordered abelian group,* $K \subseteq S$ *is an upper interval and* $S + K \subseteq K$*.*

Proposition 8.4. *If* R *is a formally real toring, then* $\mathcal{H}(R)$ *is an r-tomonoid. In particular,* $\mathcal{H}(R)$ *is formally integral.*

Proof. There a toring surjection $\phi : D \to R$, where D is a totally ordered domain. Let F be the ordered field of fractions of D. Then $\mathcal{H}(D) \subseteq \mathcal{H}(F)$, and the latter is a totally ordered group. To finish, it suffices to show that if $[\phi(x)] = [\phi(y)]$, then either $[x] = [y]$ or $[\phi(y)] = \infty$. Suppose $x, y \in D^+$, $[x] < [y]$ and $\phi(y) \neq 0$. Then $x > ny$ for all $n \in \mathbf{Z}$, and $x > z$ for all $z \in \ker \phi$. It follows that $[\phi(x)] < [\phi(y)]$. ▪

Not every formally integral Hion tomonoid is an r-tomonoid. Let U be the quotient of $\langle 9, 12, 16\rangle/33$ obtained by identifying 30 and 32. Using a, b and c to denote 9, 12 and 16, respectively, we have:

$$U = \{\, 0 < a < b < c < 2a < a + b < 2b < a + c < 3a < b + c < 2a + b = 2c < \infty \,\}.$$

Then U is a formally intgral Hion tomonoid but it is not an r-tomonoid, since $2a + b < 2c$ in any totally ordered group in which $2b < a + c$ and $3a < b + c$.

It is not hard to see that if H is an r-tomonoid, then for any totally ordered domain A, $A(H)$ is formally real. However, we have not been able to answer the following:

Problem 5. *Is the converse of 8.4 true—that is, if* $\mathcal{H}(R)$ *is an r-tomonoid, does it follow that* R *is formally real?*

9. Implementation of a search algorithm

Section not yet written.

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