TAYLOR COEFFICIENTS OF MOCK-JACOBI FORMS AND MOMENTS OF PARTITION STATISTICS

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Abstract. We develop a new technique for deriving asymptotic series expansions for moments of combinatorial generating functions that uses the transformation theory of Jacobi forms and "mock" Jacobi forms, as well as the Hardy-Ramanujan Circle Method. The approach builds on a suggestion of Zagier, who observed that the moments of a combinatorial statistic can be simultaneously encoded as the Taylor coefficients of a function that transforms as a Jacobi form. Our use of Jacobi transformations is a novel development in the subject, as previous results on the asymptotic behavior of the Taylor coefficients of Jacobi forms have involved the study of each such coefficient individually using the theory of quasimodular forms and quasimock modular forms.

As an application, we find asymptotic series for the moments of the partition rank and crank statistics. Although the coefficients are exponentially large, the error in the series expansions is polynomial, and have the same order as the coefficients of the residual Eisenstein series that are undetectable by the Circle Method. We also prove asymptotic series expansions for the symmetrized rank and crank moments introduced by Andrews and Garvan, respectively. Equivalently, the former gives asymptotic series for the enumeration of Andrews k-marked Durfee symbols.

1. Introduction and Statement of Results

In [1], Andrews defined the smallest parts function $\mathrm{spt}(n)$ as the sum of the total number of appearances of the smallest part in each integer partition of n. For example if $n = 4$, then $\mathrm{spt}(4) = 10$, since

$$4, \qquad 3+1, \qquad 2+2, \qquad 2+1+1, \qquad 1+1+1+1$$

are the partitions of 4, and they have smallest parts $4, 1, 2, 1$, and 1, respectively. Andrews related $\mathrm{spt}(n)$ to Dyson's rank of the partitions of n (see (1.1) for the definition) through the formula

$$\mathrm{spt}(n) = np(n) - \frac{1}{2} N_2(n),$$

where $N_2(n)$ is the second rank moment n (defined in (1.4)) and $p(n)$ is the number of partitions of n. Alternatively, the spt-function may be written as the difference of second moments, namely,

$$\mathrm{spt}(n) = \frac{1}{2}\left(M_2(n) - N_2(n)\right).$$

This follows from Dyson's identity $np(n) = \frac{1}{2} M_2(n)$ (see [18]), where $M_2(n)$ is the second moment of the crank statistic (using the definitions in (1.2) and (1.3)).

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These observations have been the catalyst for many works studying the relations between the rank and crank statistics (see [7, 9, 21, 22]). For instance, since $\operatorname{spt}(n)$ is positive we have the strong inequality

$$M_2(n) > N_2(n).$$

Additionally, the first two authors previously showed that as n goes to infinity

$$N_2(n), M_2(n) \sim \frac{1}{2\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}},$$

whereas the difference of these moments has (polynomially) lower asymptotic order, as

$$\operatorname{spt}(n) \sim \frac{1}{\pi\sqrt{2n}} e^{\pi\sqrt{\frac{2n}{3}}}$$

(see [7] and [10]). The fact that the main asymptotic terms of the moments match exactly is notable because it suggests that the rank and crank generating functions are more closely related than one would expect from the definitions of the statistics. The primary purpose of this paper is to explain how these types of asymptotic relationships follow from the fact that the underlying generating functions for the partitions statistics are essentially examples of Jacobi forms and "mock" Jacobi forms.

Jacobi forms should be understood as two variable automorphic forms satisfying an elliptic transformation and a modular transformation. Such functions were introduced and studied by Eichler and Zagier; see [19] for further background and applications. Key features of Jacobi forms include the fact that the specialization of the elliptic variable to a root of unity yields a modular form, whereas the Taylor coefficients with respect to the elliptic variable are each *quasimodular* forms in the modular variable. Such functions are essentially defined to be linear combinations of derivatives of modular forms, which also have well-understood transformation properties.

In his seminal thesis on mock theta functions [29], Zwegers also initiated the study of *real-analytic* Jacobi forms, which are generalizations of classical Jacobi forms. These are non-holomorphic functions of two complex variables that transform in the same way as Eichler and Zagier's Jacobi forms (see Proposition 2.5). In this case, if the elliptic variable of one of Zwegers' functions is specialized to a root of unity, then the resulting holomorphic part is a *mock modular form*, which is now known to be related to the theory of harmonic weak Maass forms (see [16] for more on the development of this theory, and [13, 28] for examples). In particular, Zwegers' work was motivated by Ramanujan's famous mock theta functions, and the mock Jacobi forms helped provide a theoretical basis for Ramanujan's observations. Furthermore, the Taylor coefficients of these functions are *quasimock modular forms* (following [9]), which are linear combinations of derivatives of mock modular functions, and whose modular transformations are again understood. The real-analytic Jacobi forms studied by Zwegers also fit into the more general framework of *mock Jacobi forms* introduced by the first author and Richter in [14].

In previous work in the subject, the cuspidal asymptotic properties of quasimodular and quasimock modular forms have only been studied through the action of modular transformations on each individual function. In contrast, the method employed in this paper directly utilizes Jacobi transformations in order to simultaneously study all of the Taylor coefficients. As a consequence, our new approach will show that the nearly identical asymptotics of moments of the rank and crank statistics arise from the fact that the Jacobi forms involved satisfy Jacobi transformations of different index.

Much of this subject can trace its motivation to Ramanujan's famous congruences in [25], which state that for $\ell \in \{5, 7, 11\}$, the partition function satisfies the linear congruence $p(\ell n + \delta_\ell) \equiv 0 \pmod{\ell}$ for all $n \geq 0$, where δ_ℓ is defined as the minimal positive residue of $\overline{24} \pmod{\ell}$. In an effort to provide a combinatorial explanation of Ramanujan's congruences Dyson introduced [17] the *rank* of a partition λ, defined as

$$\operatorname{rank}(\lambda) := \text{largest part of } \lambda - \text{number of parts of } \lambda. \tag{1.1}$$

Dyson observed that the rank is not sufficient to decompose all three of Ramanujan's congruences, and thus he also conjectured the existence of an analogous statistic, the "crank", that would explain all of Ramanujan's congruences simultaneously. Garvan found the crank statistic for vector partitions [20], and together with Andrews presented the following definition [4]. Let $o(\lambda)$ denote the number of ones in λ, and define $\mu(\lambda)$ as the number of parts strictly larger than $o(\lambda)$. Then

$$\operatorname{crank}(\lambda) := \begin{cases} \text{largest part of } \lambda & \text{if } o(\lambda) = 0, \\ \mu(\lambda) - o(\lambda) & \text{if } o(\lambda) > 0. \end{cases} \tag{1.2}$$

Let $\mathcal{M}(m,n)$ (resp. $\mathcal{N}(m,n)$) be the number of partitions of n with crank (resp. rank) m. Then aside from the anomalous case of $\mathcal{M}(m,n)$ when $n = 1$ (where the correct values are $\mathcal{M}(0,1) = 1$ and $\mathcal{M}(m,1) = 0$ for all $m \neq 0$), the two-parameter generating functions may be written as [4, 6]

$$C(x;q) := \sum_{\substack{m\in\mathbb{Z}\\ n\geq 0}} \mathcal{M}(m,n)x^m q^n = \prod_{n\geq 1} \frac{1-q^n}{(1-xq^n)(1-x^{-1}q^n)} = \frac{1-x}{(q)_\infty} \sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{n(n+1)/2}}{1-xq^n},$$

$$R(x;q) := \sum_{\substack{m\in\mathbb{Z}\\ n\geq 0}} \mathcal{N}(m,n)x^m q^n = \sum_{n\geq 0} \frac{q^{n^2}}{(xq;q)_n(x^{-1}q;q)_n} = \frac{1-x}{(q)_\infty} \sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{n(3n+1)/2}}{1-xq^n}.$$

where $(x;q)_n = (x)_n := \prod_{j=0}^{n-1}\left(1-xq^j\right)$ for all $n \in \mathbb{N}_0 \cup \{\infty\}$. For a nonnegative integer k, define the k-th *crank moment* as

$$M_k(n) := \sum_{m\in\mathbb{Z}} m^k\, \mathcal{M}(m,n), \tag{1.3}$$

and the k-th *rank moment* as

$$N_k(n) := \sum_{m\in\mathbb{Z}} m^k\, \mathcal{N}(m,n). \tag{1.4}$$

Due to the symmetries of the statistics, the crank and rank moments vanish when k is odd [5].

A vast generalization of the remarks concerning the spt-function was proposed by Garvan in [21]. In particular, he conjectured that for all $k \geq 1$ and $n \geq 1$

$$M_{2k}(n) > N_{2k}(n). \tag{1.5}$$

In a groundbreaking recent paper, Garvan himself proved this conjecture by constructing explicit higher order spt-functions [22]. In particular, this gives a precise combinatorial description for the objects that are enumerated by $M_{2k}(n) - N_{2k}(n)$ (or their symmetrized versions, see below), and the moment inequalities follow directly from the positivity of these enumerations.

We consider asymptotic results for the rank and crank moments. The cases $k = 2$ and 3 of Garvan's Conjecture were previously proven by the first two authors [10] for sufficiently large n. As is the case for $k = 1$, they proved that $M_{2k}(n) \sim N_{2k}(n)$ as $n \to \infty$ for $k = 2$ and $k = 3$, and conjectured this to be true for all $k \geq 1$. All three authors then used an extension of this method in [12] in order to prove Garvan's Conjecture for any fixed k and sufficiently large n. The approach relied on the Circle Method, identities for multiple sums of Bernoulli numbers, and the theory of quasimock theta functions. In particular, the rank moments were related to the crank moments through Atkin and Garvan's rank-crank PDE [5].

The rank-crank PDE is a relation between a triple sum of products of crank moment generating functions, rank moment generating functions, and q-derivatives of rank generating functions. Atkin and Garvan [5] had previously shown that the crank moment generating functions are quasimodular forms. Along with Garvan, the first two authors used the rank-crank PDE to connect the automorphic properties of ranks to cranks, and showed that the rank moment generating functions are quasimock theta functions [9]. Consequently, these moment generating functions have automorphic properties, which were exploited in order to prove congruences and asymptotics for their coefficients. The first author and Zwegers [15, 30] further showed that this PDE may be understood via the action of the heat operator on real-analytic Jacobi forms. However, our present approach circumvents the use of the rank-crank PDE, as we instead work directly with the Jacobi transformations of the two-variable generating series.

Let

$$C_k(q) := \sum_{n\geq 0} M_k(n)q^n \qquad \text{and} \qquad R_k(q) := \sum_{n\geq 0} N_k(n)q^n$$

denote the generating functions for $M_k(n)$ and $N_k(n)$. Let

$$\mathcal{C}(u;q) := \sum_{k=0}^{\infty} C_k(q)\frac{(2\pi iu)^k}{k!} \qquad \text{and} \qquad \mathcal{R}(u;q) := \sum_{k=0}^{\infty} R_k(q)\frac{(2\pi iu)^k}{k!}$$

be the exponential generating functions for the crank and rank moment generating functions. Rearranging the order of summation yields

$$\mathcal{C}(u;q) = C\left(e^{2\pi iu};q\right) \qquad \text{and} \qquad \mathcal{R}(u;q) = R\left(e^{2\pi iu};q\right). \tag{1.6}$$

We determine the Taylor expansion for $\mathcal{R}(u;q)$ in u, calculate the asymptotics for the individual coefficients with $q = e^{\frac{2\pi i}{k}(h+iz)}$, and finally use the Circle Method to obtain asymptotics for the moments. It is important to note that the transformation properties and asymptotics for the rank generating function are quite intricate due to its mock modular behavior.

In order to state our asymptotic formulas, we introduce some additional notation. If a, b, and c are nonnegative integers, let

$$\kappa(a,b,c) := \frac{(2(a+b+c))!}{a!(2b+1)!(2c)!}\frac{(-1)^{a+c}}{\pi^a 4^{a+b}}B_{2c}\left(\frac{1}{2}\right), \tag{1.7}$$

where $B_n(x)$ denotes the nth Bernoulli polynomial, which is defined by the generating function (3.1). Moreover, define the Kloosterman sum

$$K_k(n) := \sum_{\substack{0\leq h<k \\ (h,k)=1}} \omega_{h,k} e^{-\frac{2\pi ihn}{k}}, \tag{1.8}$$

where $\omega_{h,k}$ is the multiplier of the partition generating function, which is given explicitly by

$$\omega_{h,k} := \exp\left(\pi i s(h,k)\right). \tag{1.9}$$

Here we follow the standard notation for Dedekind sums, namely

$$s(h,k) := \sum_{\mu \pmod k} \left(\left(\frac{\mu}{k}\right)\right)\left(\left(\frac{h\mu}{k}\right)\right),$$

with the sawtooth function defined as

$$((x)) := \begin{cases} x - \lfloor x\rfloor - \frac{1}{2} & \text{if } x \in \mathbb{R}\setminus\mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases}$$

We prove the following asymptotic series expansion for the rank and crank moments.

Theorem 1.1. *For all ℓ and n we have*

$$M_{2\ell}(n) = 2\pi \sum_{k\leq\sqrt{n}} \frac{K_k(n)}{k} \sum_{a+b+c=\ell} k^a \kappa(a,b,c)(24n-1)^{c+\frac{a}{2}-\frac{3}{4}} I_{\frac{3}{2}-2c-a}\left(\frac{\pi\sqrt{24n-1}}{6k}\right) + O\left(n^{2\ell-1}\right)$$

and

$$N_{2\ell}(n) = 2\pi \sum_{k\leq\sqrt{n}} \frac{K_k(n)}{k} \sum_{a+b+c=\ell} (3k)^a \kappa(a,b,c)(24n-1)^{c+\frac{a}{2}-\frac{3}{4}} I_{\frac{3}{2}-2c-a}\left(\frac{\pi\sqrt{24n-1}}{6k}\right) + O\left(n^{2\ell-1}\right),$$

where I_ν denotes the modified Bessel function of order ν.

Remark. Here and throughout the rest of the paper, the sums on a, b, and c are only for nonnegative integers.

Remark. Using $I_\nu(y) \sim \frac{1}{\sqrt{2\pi y}} e^y$ and $y \to \infty$ one sees that

$$N_{2\ell}(n) \sim M_{2\ell}(n) \sim 2\sqrt{3}(-1)^\ell B_{2\ell}\left(\frac{1}{2}\right)(24n)^{\ell-1} e^{\pi\sqrt{\frac{2n}{3}}}.$$

Remark. In the approach used in this paper, the Bernoulli numbers $B_{2\ell}(\frac{1}{2})$ arise from the Fourier expansion of $\frac{1}{\sinh(x)}$. On the other hand, in [12] they appear as a convolution of the usual Bernoulli numbers $B_{2i} = B_{2i}(0)$.

In [1], Andrews presents the computation of the asymptotics of the symmetrized rank moments $\eta_{2k}(n)$ as a problem of significant interest. Furthermore, Garvan subsequently introduced symmetrized crank moments $\nu_{2k}(n)$ in his study of generalized spt-functions [22]. See (4.2) and (4.3) for precise definitions. Theorem 1.1 easily leads to precise asymptotics for these moments.

Corollary 1.2. *Assuming the notation of Theorem 1.1, for all ℓ and n we have*

$$\eta_{2\ell}(n) = \frac{2\pi}{(2\ell)!} \sum_{k\leq\sqrt{n}} \frac{K_k(n)}{k} \sum_{a+b+c\leq\ell} t(\ell, a+b+c)(3k)^a \kappa(a,b,c)(24n-1)^{c+\frac{a}{2}-\frac{3}{4}} I_{\frac{3}{2}-2c-a}\left(\frac{\pi\sqrt{24n-1}}{6k}\right) + O\left(n^{2\ell-1}\right),$$

and

$$\nu_{2\ell}(n) = \frac{2\pi}{(2\ell)!} \sum_{k\leq\sqrt{n}} \frac{K_k(n)}{k} \sum_{a+b+c\leq\ell} t(\ell, a+b+c) k^a \kappa(a,b,c)(24n-1)^{c+\frac{a}{2}-\frac{3}{4}} I_{\frac{3}{2}-2c-a}\left(\frac{\pi\sqrt{24n-1}}{6k}\right) + O\left(n^{2\ell-1}\right),$$

where $t(\ell, m)$ is defined by the generating function $\sum_{m=0}^{\ell} t(\ell,m)x^{2m} := \prod_{j=0}^{\ell-1}(x^2 - j^2)$.

Remark. The case $\ell = 1$ was given in [7]. Exact formulas for $\eta_{2\ell}(n)$ remain out of reach due to the fact that the moment generating functions are comprised of terms with positive weight as (mock) modular forms (specifically, as large as $2\ell - 1/2$). In such situations the Circle Method or spectral techniques cannot detect the contribution from holomorphic or cuspidal modular forms.

Remark. Andrews [1] stated the problem of asymptotics for $\eta_{2\ell}(n)$ in the context of a family of generalized partition-type combinatorial objects known as *k-marked Durfee symbols.* Letting $\mathcal{D}_k(n)$ denote the number of k-marked Durfee symbols, Andrews proved that $\mathcal{D}_k(n) = \eta_{2(k-1)}(n)$.

The remainder of the paper is structured as follows. In Section 2 we build upon Zwegers' results in order to derive Jacobi transformation formulas for the crank and rank generating functions, and we also record some useful identities for the corresponding multiplier systems. We next study the crank and rank moments as Taylor coefficients in Section 3, and provide bounds for their asymptotic behavior. Finally, we conclude by applying the Hardy-Ramanujan Circle Method in Section 4 in order to prove Theorem 1.1 and Corollary 1.2.

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2. Transformation Properties

In this section we derive transformation properties for the crank and rank generating functions. In his study of mock theta functions [28], Zwegers proved transformation formulas for the completed automorphic forms, but he did not find simplified transformations for the mock modular forms themselves. It is therefore necessary for us to carefully study these transformations, and we use properties of Fourier-Whittaker coefficient expansions in order to find cancellations in the non-holomorphic terms.

Throughout, we let $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$ and $0 \leq h \leq k$ with $(h,k) = 1$. We write $x = e^{2\pi i u}$ and $q = e^{-2\pi z}$. Moreover we denote by $[a]_b$ the inverse of a modulo b, and we allow this notation to (implicitly) extend to higher moduli, and also (implicitly) assume any legal divisibility properties. In other words, $[a]_b$ can be replaced by $[a]_{bc}$ for any c such that $(a,c) = 1$, and we may assume $d \mid [a]_b$ for any d such that $(b,d) = 1$. We will also make the explicit choices that $3 \mid [\cdot]_k$ if $3 \nmid k$, and use $[\cdot]_{3k}$ instead of $[\cdot]_k$ and $[\cdot]_{\frac{k}{3}}$ if $3 \mid k$, and also $[\cdot]_{8k}$ instead of $[\cdot]_k$ if k is even, and $8 \mid [\cdot]_k$ if k is odd. To avoid ambiguity, this will be further clarified in the text that follows.

2.1. **Cranks.** We easily see

$$C(x;q) = \frac{(q)_\infty}{(xq)_\infty (x^{-1}q)_\infty} = -\frac{2\sin(\pi u) q^{\frac{1}{24}} \eta^2(iz)}{\vartheta(u;iz)}, \tag{2.1}$$

where the Dedekind eta function and the Jacobi theta function are defined by

$$\eta(iz) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n),$$

$$\vartheta(u;iz) := \sum_{\nu \in \mathbb{Z}+\frac{1}{2}} e^{-\pi \nu^2 z + 2\pi i \nu \left(u+\frac{1}{2}\right)} = -iq^{\frac{1}{8}} x^{-\frac{1}{2}} \prod_{n=1}^{\infty} (1-q^n)\left(1-xq^{n-1}\right)\left(1-x^{-1}q^n\right). \tag{2.2}$$

Define $\chi(h,[-h]_k,k)$ to be the multiplier of the Dedekind eta-function, so that

$$\eta\left(\frac{1}{k}(h+iz)\right) = \sqrt{\frac{i}{z}} \chi\Big(h,[-h]_k,k\Big) \eta\left(\frac{1}{k}\left([-h]_k + \frac{i}{z}\right)\right). \tag{2.3}$$

It is known that [3]

$$\chi(h,[-h]_k,k) := \begin{cases} \left(\frac{h}{k}\right) i^{-\frac{k}{2}} e^{\frac{\pi i}{12}\left(-\beta[-h]_k\left(1-k^2\right)+k(h-[-h]_k)\right)} & \text{if } k \text{ is odd,} \\ i^{-\frac{1}{2}} \left(\frac{k}{h}\right) e^{\frac{\pi i}{12}\left(hk\left(1-[-h]_k^2\right)-[-h]_k(\beta-k+3)\right)} & \text{if } h \text{ is odd,} \end{cases} \tag{2.4}$$

where β is defined by $-h[-h]_k - \beta k = 1$.

Remark 2.1. In the notation of Section 1 we have

$$\chi\left(h,[-h]_k,k\right) = i^{-\frac{1}{2}}\omega_{h,k}^{-1}e^{-\frac{\pi i}{12k}([-h]_k-h)}.$$

Note that we must be careful when picking the representative of the inverse of $-h$ modulo k, as $\chi(h,[-h]_k,k)$ depends on the choice of $[-h]_k$ modulo $\mathrm{lcm}(24,k)$.

Moreover we will require the following transformation law for ϑ from [26]:

$$\vartheta\left(u;\frac{1}{k}\left(h+iz\right)\right) = \sqrt{\frac{i}{z}}\chi^3\Big(h,[-h]_k,k\Big)e^{-\frac{\pi k u^2}{z}}\vartheta\left(\frac{iu}{z};\frac{1}{k}\left([-h]_k+\frac{i}{z}\right)\right). \tag{2.5}$$

Inserting (2.3) and (2.5) into (2.1), we conclude a transformation law for the crank generating function.

Proposition 2.2. *We have*

$$C\left(e^{2\pi i u};e^{\frac{2\pi i}{k}(h+iz)}\right) = -2\sin(\pi u)\chi^{-1}\Big(h,[-h]_k,k\Big)\sqrt{\frac{i}{z}}e^{\frac{\pi i}{12k}(h+iz)}\frac{\eta^2\left(\frac{1}{k}\left([-h]_k+\frac{i}{z}\right)\right)}{\vartheta\left(\frac{iu}{z};\frac{1}{k}\left([-h]_k+\frac{i}{z}\right)\right)}e^{\frac{\pi k u^2}{z}}.$$

Remark. By rewriting the right-hand side of Proposition 2.2 in terms of the crank generating function, one sees that $C(x;q)$ satisfies Jacobi transformations of weight $\frac{1}{2}$ and index $-\frac{1}{2}$.

2.2. **Lerch sums and Zwegers' thesis.** In his landmark thesis [29], Zwegers constructed an infinite family of so-called harmonic weak Maass forms by "completing" certain Lerch sums. To state his results, first define the function

$$\mu(u,v) = \mu(u,v;iz) := \frac{x^{\frac{1}{2}}}{\vartheta(v;iz)}\sum_{n\in\mathbb{Z}}\frac{(-w)^n q^{\frac{n(n+1)}{2}}}{1-xq^n}, \tag{2.6}$$

where $x := e^{2\pi i u}$, $w := e^{2\pi i v}$, and $q := e^{-2\pi z}$. Zwegers proved that this function can be completed to a non-holomorphic automorphic form. The non-holomorphic correction factor requires the definition

$$S(u) = S(u;iz) := \sum_{\nu\in\mathbb{Z}+\frac{1}{2}}(-1)^{\nu-\frac{1}{2}}\left\{\mathrm{sgn}(\nu) - E\left(\left(\nu+\frac{\mathrm{Im}(u)}{\mathrm{Im}(iz)}\right)\sqrt{2\mathrm{Im}(iz)}\right)\right\}e^{-2\pi i\nu u}q^{-\nu^2/2}. \tag{2.7}$$

Here $E(x)$ is defined by

$$E(x) := 2\int_0^x e^{-\pi u^2}du = \mathrm{sgn}(x)\left(1-\beta\left(x^2\right)\right),$$

where for positive real x we let $\beta(x) := \int_x^\infty u^{-\frac{1}{2}}e^{-\pi u}du$. Zwegers [29] proved several useful transformation properties of S.

Proposition 2.3. *If $u \in \mathbb{C}$ and $Re(z) > 0$, then*

(*i*) $S(u+1;iz) = -S(u;iz)$,
(*ii*) $S(u;iz+1) = e^{-\frac{\pi i}{4}}S(u;iz)$,
(*iii*) $S(u;iz) = -\frac{1}{\sqrt{z}}e^{\frac{\pi u^2}{z}}\left(S\left(\frac{iu}{z};\frac{i}{z}\right) - H\left(\frac{iu}{z};\frac{i}{z}\right)\right)$,

where the Mordell integral is defined by

$$H(u;iz) := \int_{-\infty}^{\infty}\frac{e^{-\pi z x^2-2\pi x u}}{\cosh(\pi x)}dx.$$

Moreover, we need the following "dissection" property of S proved by the first author and Folsom.

Proposition 2.4 (Proposition 2.3 of [8])**.** *For $n \in \mathbb{N}$, we have*

$$S\left(u;\frac{iz}{n}\right)=\sum_{\ell=0}^{n-1} q^{-\frac{1}{2n}\left(\ell-\frac{n-1}{2}\right)^2} e^{-2\pi i\left(\ell-\frac{n-1}{2}\right)\left(u+\frac{1}{2}\right)} S\left(nu+\left(\ell-\frac{n-1}{2}\right)iz+\frac{n-1}{2};niz\right).$$

Zwegers defined the real analytic function

$$\widehat{\mu}(u,v)=\widehat{\mu}(u,v;iz):=\mu(u,v;iz)+\frac{i}{2}S(u-v;iz),$$

which should be considered as the modular "completion" of μ. This function satisfies the following elliptic and modular transformation laws.

Proposition 2.5. *Assume all of the notation and hypotheses from above. If $k,\ell,m,n\in\mathbb{Z}$, then we have*

(i) $\widehat{\mu}(u+kiz+\ell,v+miz+n)=(-1)^{k+\ell+m+n}e^{-\pi z(k-m)^2+2\pi i(k-m)(u-v)}\widehat{\mu}(u,v)$,

(ii) $\widehat{\mu}\left(-iuz,-ivz;\frac{1}{k}\left(h+iz\right)\right)=\chi^{-3}(h,[-h]_k,k)\sqrt{\frac{i}{z}}e^{-\pi kz(u-v)^2}\widehat{\mu}\left(u,v;\frac{1}{k}\left([-h]_k+\frac{i}{z}\right)\right)$.

2.3. **Ranks.** Here we prove a modular transformation formula for the rank generating function. To state these we require the following multipliers

$$\begin{aligned}
\xi_\ell(h,k) &:= (-1)^{\ell+1}e^{-\frac{\pi ih}{4k}(2\ell+1)^2\mp\frac{\pi i}{k}\left(\frac{\widetilde{h}-h}{3}\right)(2\ell+1)+\frac{2\pi i\widetilde{h}}{9k}},\\
\xi(h,k) &:= e^{\frac{\pi i}{4}}\chi^{-3}\Big(h,[-h]_k,k\Big)(-1)^{\widetilde{h}+h}e^{\frac{2\pi i\widetilde{h}}{9k}-\frac{\pi i[-h]_k}{k}\left(\frac{\widetilde{h}-h}{3}\right)^2},\\
\xi'(h,k) &:= (-1)^k\xi(h,k).
\end{aligned}$$

Here $\widetilde{h}\in\{-1,0,1\}$ is defined by $\widetilde{h}\equiv h \pmod 3$. If $0\le\ell\le k-1$, then we also write $\alpha^{\pm}(\ell,k):=\frac{1}{k}\left(\pm\frac{1}{3}-\left(\ell-\frac{k-1}{2}\right)\right)$. Note that in all cases $|\alpha^{\pm}(\ell,k)|<\frac{1}{2}$.

Proposition 2.6. (i) *For $3\mid k$, we have*

$$\begin{aligned}
R\Big(e^{2\pi iu};e^{\frac{2\pi i}{k}(h+iz)}\Big) =& -\frac{2\sin(\pi u)}{\sqrt{z}}e^{-\frac{\pi z}{12k}-\frac{\pi ih}{4k}+\frac{3\pi ku^2}{z}+\frac{\pi}{3kz}}\\
&\times\sum_{\pm}\pm\Bigg[\xi'\left(h,\frac{k}{3}\right)e^{\mp\frac{2\pi\widetilde{h}u}{z}}\mu\left(\frac{3iu}{z},\pm\frac{\widetilde{h}}{k}\left([-h]_{3k}+\frac{i}{z}\right);\frac{3}{k}\left([-h]_{3k}+\frac{i}{z}\right)\right)\\
&+\frac{i}{2}\sqrt{\frac{3}{k}}\sum_{\ell=0}^{\frac{k}{3}-1}\xi_\ell\left(h,\frac{k}{3}\right)H\left(\frac{3iu}{z}+\alpha^{\pm}\left(\ell,\frac{k}{3}\right)\mp\frac{\widetilde{h}i}{kz};\frac{3i}{kz}\right)\Bigg]\\
&-2\sqrt{\frac{i}{z}}e^{\frac{3\pi ku^2}{z}}\chi^{-1}\Big(h,[-h]_{3k},k\Big)\frac{\sin(\pi u)e^{\frac{\pi i}{12k}(h+iz)}\eta^3\left(\frac{3}{k}\left([-h]_{3k}+\frac{i}{z}\right)\right)}{\eta\left(\frac{1}{k}\left([-h]_{3k}+\frac{i}{z}\right)\right)\vartheta(\frac{3iu}{z};\frac{3}{k}([-h]_{3k}+\frac{i}{z}))}.
\end{aligned}$$

(ii) For $3 \nmid k$, we have

$$R\left(e^{2\pi iu}; e^{\frac{2\pi i}{k}(h+iz)}\right) = -\frac{2\sin(\pi u)}{\sqrt{3z}} e^{-\frac{\pi z}{12k} - \frac{\pi ih}{4k} + \frac{3\pi ku^2}{z}} \sum_{\pm} \pm\Bigg[\xi(3h,k)\mu\left(\frac{iu}{z}, \mp\frac{k}{3}; \frac{1}{3k}\left([-h]_k + \frac{i}{z}\right)\right)$$
$$+ \frac{i}{2\sqrt{k}} \sum_{\ell=0}^{k-1} \xi_\ell(3h,k) H\left(\frac{iu}{z} + \alpha^{\pm}(\ell,k); \frac{i}{3kz}\right)\Bigg]$$
$$- \frac{2}{3}\sqrt{\frac{i}{z}} e^{\frac{3\pi ku^2}{z}} \chi^{-1}\Big(h, [-h]_k, k\Big) \frac{\sin(\pi u) e^{\frac{\pi i}{12k}(h+iz)} \eta^3\left(\frac{1}{3k}\left([-h]_k + \frac{i}{z}\right)\right)}{\eta\left(\frac{1}{k}\left([-h]_k + \frac{i}{z}\right)\right) \vartheta\left(\frac{iu}{z}; \frac{1}{3k}\left([-h]_k + \frac{i}{z}\right)\right)}.$$

Remark. The key feature of this result is that the error terms in the transformations only involve the Mordell integrals H; there are no non-holomorphic S-terms.

Remark. The shape of the initial multiplicative factor in the formulas of Proposition 2.6 means that the (completed) rank generating function satisfies Jacobi transformations of weight $\frac{1}{2}$ and index $-\frac{3}{2}$.

To prove Proposition 2.6, we use the identity (see, for instance, [27])

$$R(x;q) = -i(1-x)x^{-\frac{1}{2}}q^{-\frac{1}{8}}\left(-x^{-1}\mu(3u,-iz;3iz) + x\mu(3u,iz;3iz) + q^{\frac{1}{6}}\frac{\eta^3(3iz)}{\eta(iz)\vartheta(3u;3iz)}\right) \tag{2.8}$$

and determine the transformation law of the individual components. We begin with a related transformation law for $\mu\left(u, \frac{1}{3}iz; iz\right)$.

Proposition 2.7. *Using the notation above, we have*

$$\mu\left(u, \pm\frac{1}{3k}(h+iz); \frac{1}{k}(h+iz)\right) = \frac{1}{\sqrt{z}} e^{-\frac{\pi z}{9k} + \frac{\pi k}{z}\left(u \mp \frac{\widetilde{h}}{3k}\right)^2 \mp \frac{2\pi iu}{3}}$$
$$\times\Bigg[\xi(h,k)\mu\left(\frac{iu}{z}, \pm\frac{\widetilde{h}}{3k}\left([-h]_k + \frac{i}{z}\right) \mp \frac{1}{3k}\Big(1 + h[-h]_k\Big); \frac{1}{k}\left([-h]_k + \frac{i}{z}\right)\right)$$
$$+ \frac{i}{2\sqrt{k}} \sum_{\ell=0}^{k-1} \xi_\ell(h,k) H\left(\frac{iu}{z} \mp \frac{\widetilde{h}i}{3kz} + \alpha^{\pm}(\ell,k); \frac{i}{kz}\right)\Bigg].$$

Proof. We decompose μ into $\widehat{\mu}$ and S-terms and use transformation laws for these components. We start with the function S. Note that to return to the desired identity we must multiply all occurrences of S by $-\frac{i}{2}$. Using Proposition 2.4 we have

$$S\left(u \mp \frac{1}{3k}(h+iz); \frac{1}{k}(h+iz)\right) = \sum_{\ell=0}^{k-1} e^{-\frac{\pi i}{k}\left(\ell - \frac{k-1}{2}\right)^2(h+iz) - 2\pi i\left(u \mp \frac{1}{3k}(h+iz) + \frac{1}{2}\right)\left(\ell - \frac{k-1}{2}\right)} \tag{2.9}$$
$$\times S\left(ku \mp \frac{1}{3}(h+iz) + \left(\ell - \frac{k-1}{2}\right)(h+iz) + \frac{k-1}{2}; k(h+iz)\right).$$

We first use Proposition 2.3 parts (i) and (ii) to shift the arguments of the S functions, and then use Proposition 2.3 (iii) to invert the z argument. In all, we rewrite the S-terms from (2.9) as

$$(-1)^{1 + \frac{\widetilde{h}-h}{3} + \ell h + \frac{(1-h)(k-1)}{2}} e^{-\frac{\pi ihk}{4}} \frac{1}{\sqrt{kz}} e^{\frac{\pi}{kz}\left(ku \mp \frac{\widetilde{h}}{3} + iz\left(\mp\frac{1}{3} + \left(\ell - \frac{k-1}{2}\right)\right)\right)^2}$$
$$\times\left(S\left(\frac{iu}{z} \mp \frac{\widetilde{h}i}{3kz} + \alpha^{\pm}(\ell,k); \frac{i}{kz}\right) - H\left(\frac{iu}{z} \mp \frac{\widetilde{h}i}{3kz} + \alpha^{\pm}(\ell,k); \frac{i}{kz}\right)\right).$$

Inserting this into (2.9) gives after a lengthy but straightforward calculation that (2.9) equals

$$e^{-\frac{\pi z}{9k}+\frac{\pi}{kz}\left(\mp\frac{\widetilde{h}}{3}+ku\right)^2\mp\frac{2\pi iu}{3}}\frac{1}{\sqrt{kz}}\sum_{\ell=0}^{k-1}\xi_\ell(h,k)\Bigg(S\left(\frac{iu}{z}\mp\frac{\widetilde{h}i}{3kz}+\alpha^{\pm}(\ell,k);\frac{i}{kz}\right)\\ -H\left(\frac{iu}{z}\mp\frac{\widetilde{h}i}{3kz}+\alpha^{\pm}(\ell,k);\frac{i}{kz}\right)\Bigg).$$

We next turn to $\widehat{\mu}$. Using Proposition 2.5 (ii), we obtain

$$\begin{aligned}(2.10)\quad &\widehat{\mu}\left(u,\pm\frac{1}{3k}(h+iz);\frac{1}{k}(h+iz)\right)\\ &=\chi^{-3}\Big(h,[-h]_k,k\Big)\sqrt{\frac{i}{z}}e^{-\pi kz\left(\frac{iu}{z}\mp\frac{ih}{3kz}\pm\frac{1}{3k}\right)^2}\widehat{\mu}\left(\frac{iu}{z},\pm\frac{hi}{3kz}\mp\frac{1}{3k};\frac{1}{k}\left([-h]_k+\frac{i}{z}\right)\right).\end{aligned}$$

Applying Proposition 2.5 (i) and simplifying the exponential terms implies that (2.10) equals

$$e^{-\frac{\pi z}{9k}+\frac{\pi k}{z}\left(u\mp\frac{\widetilde{h}}{3k}\right)^2\mp\frac{2\pi iu}{3}}\frac{1}{\sqrt{z}}\xi(h,k)\widehat{\mu}\left(\frac{iu}{z},\pm\frac{\widetilde{h}}{3k}\left([-h]_k+\frac{i}{z}\right)\mp\frac{1}{3k}\Big(1+h[-h]_k\Big);\frac{1}{k}\left([-h]_k+\frac{i}{z}\right)\right).$$

The proof of Proposition 2.7 is then complete upon applying the identity

$$\begin{aligned}(2.11)\quad &\frac{e^{\frac{\pi\widetilde{h}^2}{9kz}}}{\sqrt{k}}\sum_{\ell=0}^{k-1}\xi_\ell(h,k)S\left(\frac{iu}{z}\mp\frac{\widetilde{h}i}{3kz}+\alpha^{\pm}(\ell,k);\frac{i}{kz}\right)\\ &=\xi(h,k)e^{\frac{\pi\widetilde{h}^2}{9kz}}S\left(\frac{iu}{z}\mp\frac{\widetilde{h}}{3k}\left([-h]_k+\frac{i}{z}\right)\pm\frac{1}{3k}\Big(1+h[-h]_k\Big);\frac{1}{k}\left([-h]_k+\frac{i}{z}\right)\right).\end{aligned}$$

Instead of arguing directly from the definition of S in (2.7), we prove (2.11) using an argument first employed by the first two authors in [11]. The key component of this argument is the fact that Fourier-Whittaker expansions are unique. In order to simplify the following development, we write $\tau:=\frac{i}{kz}$, $v:=\frac{iu}{z}$, $y:=\mathrm{Im}\,(\tau)$. Since all the remaining terms in the transformation law (2.10) are meromorphic functions of τ and v, it is enough to show that each term in (2.11) has a Fourier expansion equal to

$$\sum_{n\in\mathbb{Q}\setminus\{0\}}a(n)\Gamma\left(\frac{1}{2};4\pi|n|y\right)q^{-n}$$

with $\Gamma(a;x):=\int_x^\infty e^{-t}t^{a-1}dt$ the incomplete gamma function. To show this, we assume that $v\in\mathbb{R}$ and conclude the remaining cases using analytic continuation. By (2.7) each S-term occurring in (2.11) has a Fourier expansion equal to

$$\begin{aligned}(2.12)\quad &e^{-\frac{\pi i\widetilde{h}^2\tau}{9}}S\left(v\mp\frac{\widetilde{h}\tau}{3}+\alpha;\tau+\beta\right).\\ &=\sum_{\nu\in\mathbb{Z}+\frac{1}{2}}\left(\mathrm{sgn}\,(\nu)-E\left(\left(\nu\mp\frac{\widetilde{h}}{3}\right)\sqrt{2y}\right)\right)(-1)^{\nu-\frac{1}{2}}e^{-\pi i\nu^2(\tau+\beta)-2\pi i\nu\left(v\mp\frac{\widetilde{h}\tau}{3}+\alpha\right)-\frac{\pi i\widetilde{h}^2\tau}{9}}.\end{aligned}$$

for some real α and β. Using that $-\frac{1}{2} < \frac{\widetilde{h}}{3} < \frac{1}{2}$ we have

$$\operatorname{sgn}(\nu) - E\left(\left(\nu \mp \frac{\widetilde{h}}{3}\right)\sqrt{2y}\right) = \frac{\operatorname{sgn}(\nu)}{\sqrt{\pi}}\Gamma\left(\frac{1}{2}, 2\pi\left(\nu \mp \frac{\widetilde{h}}{3}\right)^2 y\right).$$

Thus (2.12) is equal to

$$\frac{1}{\sqrt{\pi}}\sum_{\nu\in\mathbb{Z}+\frac{1}{2}} \operatorname{sgn}(\nu)(-1)^{\nu-\frac{1}{2}}\Gamma\left(\frac{1}{2}; 2\pi\left(\nu \mp \frac{\widetilde{h}}{3}\right)^2 y\right) e^{-\pi i\tau\left(\nu\mp\frac{\widetilde{h}}{3}\right)^2 - \pi i\nu^2\beta - 2\pi i\nu(v+\alpha)}.$$

Combining the above gives the statement of the theorem. □

Having established Proposition 2.7 we return to the proof of Proposition 2.6.

Proof of Proposition 2.6. For notational simplicity we distinguish between the cases $3|k$ and $3\nmid k$.

We start with the case $3|k$ and use that in this case, we have that $\widetilde{h}^2 = 1$. Recall that we choose $[-h]_k$ for $[-h]_{\frac{k}{3}}$. Using (2.3) and (2.5) the last summand in (2.8) transforms directly into the last summand of part (i). For the μ-functions, we send $u \to 3u$, $k \to \frac{k}{3}$ in Proposition 2.7. Thus, shifting by $\frac{1}{k}(1 + h[-h]_k) \in \mathbb{Z}$ in the second argument in μ, and noting that $(-1)^{\frac{1}{k}(1+h[-h]_k)} = (-1)^k$ gives the claim for $3 \mid k$.

Similarly, if $3 \nmid k$ the last summand in (2.8) transforms to

$$-\frac{2}{3}\sqrt{\frac{i}{z}} e^{\frac{3\pi k u^2}{z}} \chi^{-1}\Big(h, [-h]_k, k\Big) \frac{\sin(\pi u) e^{\frac{\pi i}{12k}(h+iz)} \eta^3\left(\frac{1}{k}\left([-3h]_k + \frac{i}{3z}\right)\right)}{\vartheta\left(\frac{iu}{z}; \frac{1}{k}\left([-3h]_k + \frac{i}{3z}\right)\right)\eta\left(\frac{1}{k}\left([-h]_k + \frac{i}{z}\right)\right)}.$$

This can be rewritten in the form claimed in the Proposition statement by recalling the assumptions that $3 \mid [-h]_k$ and using $[3]_{8k}$ instead of $[3]_k$.

We next turn to the terms $\mu(3u, \pm iz; 3iz)$. We send $u \mapsto 3u$, $h \mapsto 3h$, and $z \mapsto 3z$ in Proposition 2.7. Hence $\widetilde{3h} = 0$, resulting in

$$\frac{1}{\sqrt{3z}} e^{-\frac{\pi z}{3k} + \frac{3\pi k u^2}{z} \mp 2\pi i u}\Bigg(\xi(3h,k)\mu\left(\frac{iu}{z}, \mp\frac{1}{3k}\left(1 + 3h[-3h]_k\right); \frac{1}{k}\left([-3h]_k + \frac{i}{3z}\right)\right)$$
$$+ \frac{i}{2\sqrt{k}}\sum_{\ell=0}^{k-1}\xi_\ell(3h,k) H\left(\frac{iu}{z} + \alpha^{\pm}(\ell,k); \frac{i}{3kz}\right)\Bigg).$$

Using that $2 \mid [-h]_k$ for k odd and $[-3h]_{2k}$ instead of $[-3h]_k$ for k even gives

$$\mp\frac{1}{3k}\left(1 - k^2 + 3h[-3h]_k\right) \in 2\mathbb{Z}.$$

The claim now follows by using Proposition 2.3 (i) and Proposition 2.5 (i). □

2.4. **Some Multiplier Identities.** We give a pair of identities for the multipliers appearing in Proposition 2.6.

Lemma 2.8. (i) *If* $3 \mid k$*, then*

$$\xi'\left(h, \frac{k}{3}\right) = \left(\frac{h}{3}\right)\chi^{-1}\left(h, [-h]_{3k}, k\right) i^{\frac{1}{2}} e^{\frac{\pi i}{3k}(h - [-h]_{3k})}.$$

(ii) If $3 \nmid k$, *then*

$$\xi(3h,k) = i^{\frac{3}{2}}(-1)^{k+1}\left(\frac{k}{3}\right)e^{\frac{\pi i h}{3k}}\chi^{-1}\left(h,[-h]_k,k\right).$$

Proof. We begin with the case $3 \mid k$. Recall we take $[-h]_{3k}$ instead of $[-h]_{\frac{k}{3}}$. Thus

$$\xi'\left(h,\frac{k}{3}\right) = (-1)^k e^{\frac{\pi i}{4}}\chi^{-3}\left(h,[-h]_{3k},\frac{k}{3}\right)(-1)^{h+1}e^{-\frac{\pi i[-h]_{3k}}{3k}\left(1+h^2\right)}.$$

Note if $-h[-h]_{3k} - \beta k = 1$, then $3 \mid \beta$. Additionally, if $-h[-h]_{3k} - \beta'\frac{k}{3} = 1$, then $\beta' = 3\beta$. We distinguish two cases, depending on whether k is odd or even.

In the case of k odd we obtain, using that $2 \mid [-h]_{3k}$,

$$\frac{\chi(h,[-h]_{3k},k)}{\chi^3\left(h,[-h]_{3k},\frac{k}{3}\right)} = \left(\frac{h}{3}\right).$$

From this we conclude (i) for odd k.

In the case $3 \mid k$ and k is even we have $[-h]_{24k}$ instead of $[-h]_k$. Thus

$$\frac{\chi(h,[-h]_{24k},k)}{\chi^3\left(h,[-h]_{24k},\frac{k}{3}\right)} = i\left(\frac{3}{h}\right)e^{-\frac{\pi i h}{2}} = \left(\frac{h}{3}\right).$$

Noting that $e^{-\frac{\pi i h}{3k}\left(1+h[-h]_{24k}\right)} = 1$ completes the proof of (i).

We next turn to the case $3 \nmid k$. We obtain

$$\xi(3h,k) = i^{\frac{1}{2}}(-1)^h\chi^{-3}\left(3h,[-3h]_k,k\right)e^{-\frac{\pi i h^2[-3h]_k}{k}}.$$

We again consider the two cases k odd and k even separately. In the k odd case $24 \mid [-h]_k$ yielding

$$\frac{\chi\left(h,[-h]_k,k\right)}{\chi^3\left(3h,[-3h]_k,k\right)} = \left(\frac{3}{k}\right)i^k e^{-\frac{2\pi i h k}{3}}.$$

Hence

$$\xi(3h,k) = -i^{-\frac{1}{2}}\left(\frac{k}{3}\right)(-1)^h\chi^{-1}(h,[-h]_k,k)e^{-\frac{2\pi i h k}{3}-\frac{\pi i h^2[-3h]_k}{k}},$$

which leads to (ii) for k odd using that $e^{-\frac{2\pi i h k}{3}-\frac{\pi i h^2[-3h]_k}{k}} = (-1)^h e^{\frac{\pi i h}{3k}}$.

In the case $3 \nmid k$ and $2 \mid k$ we have $[-3h]_{8k}$ instead of $[-3h]_k$. Noting $1+3h[-3h]_{8k} \equiv 0 \pmod{2k}$ gives that $e^{-\frac{\pi i h}{3k}\left(1+3h[-3h]_{8k}\right)} = e^{-\frac{\pi i h k}{3}}$. From this we conclude that

$$\frac{\chi(h,[-h]_{8k},k)}{\chi^3(3h,[-3h]_{8k},k)} = i\left(\frac{k}{3}\right)e^{-\frac{2\pi i h k}{3}}.$$

Thus we obtain (ii) for $3 \nmid k$ and k even. □

3. Asymptotics of Taylor Expansions

3.1. **A General Lemma.** For $\nu \in \mathbb{R}$ define

$$f_\nu(u;z) := e^{\frac{\nu\pi u^2}{z}}\frac{\sin(\pi u)}{\sinh\left(\frac{\pi u}{z}\right)}.$$

We show the following Taylor expansion of f_ν as a function of u.

Lemma 3.1. *We have*

$$f_\nu(u;z)=\sum_{r=0}^{\infty}\frac{(2\pi iu)^{2r}}{(2r)!}\sum_{a+b+c=r}\nu^a\kappa(a,b,c)z^{1-a-2c},$$

where $\kappa(a,b,c)$ *was defined in* (1.7).

Proof. The lemma follows easily from the following Laurent series expansions:

$$e^x=\sum_{k=0}^{\infty}\frac{x^k}{k!}\quad,\qquad \sin(x)=\sum_{k=0}^{\infty}\frac{(-1)^kx^{2k+1}}{(2k+1)!},\quad\text{and}\quad \frac{1}{\sinh(x)}=x^{-1}\sum_{n=0}^{\infty}B_{2n}\left(\frac{1}{2}\right)\frac{(2x)^{2n}}{(2n)!}.$$

The last of these expansions follows from the generating function for the Bernoulli polynomials

$$\frac{te^{tx}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!},\tag{3.1}$$

using that $B_\ell\left(\frac{1}{2}\right)=0$ for ℓ odd. ☐

3.2. **Cranks.** We recall (1.6). We will use Proposition 2.2 to determine an "asymptotic" Taylor expansion as $z\to 0$. Using the product expansion in (2.2), we have

$$\vartheta\left(\frac{iu}{z};\frac{1}{k}\left([-h]_k+\frac{i}{z}\right)\right)^{-1}=i\frac{e^{-\frac{\pi i}{4k}\left([-h]_k+\frac{i}{z}\right)}}{2\sinh(\frac{\pi u}{z})}+u^{-1}\sum_{r\geq 0}a_r(z)\frac{(2\pi iu)^r}{r!}$$

with $|a_r(z)|\ll_r |z|^{1-r}e^{-\frac{7\pi}{4k}Re\left(\frac{1}{z}\right)}$. In an abuse of notation, we will repeatedly write $a_r(z)$ for the Taylor coefficients of various error terms that are polynomially or exponentially decaying as $z\to 0$.

Combining this with the asymptotic expansion of η, we obtain

$$\mathcal{C}(u;q)=-i^{\frac{3}{2}}e^{\frac{\pi i}{12k}(h-[-h]_k)}\chi^{-1}\Big(h,[-h]_k,k\Big)e^{\frac{\pi}{12k}\left(\frac{1}{z}-z\right)}z^{-\frac{1}{2}}f_k(u;z)+\sum_{r=0}^{\infty}a_r(z)\frac{(2\pi iu)^r}{r!},$$

where now $|a_r(z)|\ll |z|^{\frac{1}{2}-r}e^{-\frac{\alpha}{k}Re\left(\frac{1}{z}\right)}$ for some $\alpha>0$ independent of k (the exact value of α is unimportant here and elsewhere, and the key feature is simply that it is positive; we further abuse notation and let α denote all such exponents of error terms).

Using Lemma 3.1, we obtain an explicit asymptotic for the crank generating function.

Proposition 3.2. *In the notation above*

$$\mathcal{C}\left(u;e^{\frac{2\pi i}{k}(h+iz)}\right)=-i^{\frac{3}{2}}e^{\frac{\pi i}{12k}(h-[-h]_k)}\chi^{-1}\Big(h,[-h]_k,k\Big)e^{\frac{\pi}{12k}\left(\frac{1}{z}-z\right)}\sum_{r=0}^{\infty}\frac{(2\pi iu)^{2r}}{(2r)!}$$
$$\times\sum_{a+b+c=r}k^a\kappa(a,b,c)z^{\frac{1}{2}-a-2c}+\sum_{r=0}^{\infty}a_r(z)\frac{(2\pi iu)^r}{r!},$$

where the coefficients $a_r(z)$ *satisfy the asymptotic bounds* $a_r(z)\ll_r |z|^{\frac{1}{2}-r}e^{-\frac{\alpha}{k}Re\left(\frac{1}{z}\right)}$ *for some* $\alpha>0$ *independent of* k.

Proposition 3.2 allows us to determine the asymptotic behavior of the crank moments.

Corollary 3.3. *We have*

$$C_{2\ell}\left(e^{\frac{2\pi i}{k}(h+iz)}\right)=-i^{\frac{3}{2}}e^{\frac{\pi i}{12k}(h-[-h]_k)}\chi^{-1}\Big(h,[-h]_k,k\Big)e^{\frac{\pi}{12k}\left(\frac{1}{z}-z\right)}\sum_{a+b+c=\ell}k^a\kappa(a,b,c)z^{\frac{1}{2}-a-2c}+a_{2\ell}(z),$$

where the coefficients $a_{2\ell}(z)$ satisfy the asymptotic bounds $a_{2\ell}(z) \ll_\ell z^{\frac{1}{2}-2\ell} e^{-\frac{\alpha}{k} Re\left(\frac{1}{z}\right)}$ for some $\alpha > 0$ independent of k.

3.3. **Mordell Integrals.** In this section we provide an estimate for the H-integrals appearing in Proposition 2.6. We assume throughout the rest of the paper that $\mathrm{Re}\left(\frac{1}{z}\right) > \frac{k}{2}$. Define for $-\frac{1}{2} < \alpha < \frac{1}{2}$ the function

$$H_{k,h,\alpha}(u; z) := H\left(\frac{iu}{z} + \alpha \mp \frac{\widetilde{h}i}{3kz}; \frac{i}{kz}\right) = \int_{\mathbb{R}} \frac{e^{-\frac{\pi x^2}{kz} - 2\pi x\left(\frac{iu}{z} + \alpha \mp \frac{\widetilde{h}i}{3kz}\right)}}{\cosh(\pi x)} dx.$$

In this section we establish bounds for the Taylor coefficients of $H_{k,h,\alpha}$.

Lemma 3.4. *With $H^{(\ell)}_{k,h,\alpha}(u; z) := \left(\frac{\partial}{\partial u}\right)^\ell H(u; z)$ we have*

$$\left|H^{(\ell)}_{k,h,\alpha}(0; z)\right| \ll_\ell |z|^{-\ell}\, e^{-\frac{\pi \widetilde{h}^2}{9k} Re\left(\frac{1}{z}\right)}.$$

Proof. We rewrite

$$\begin{aligned} H_{k,h,\alpha}(u; z) &= e^{-\frac{\pi \widetilde{h}^2}{9kz} \mp \frac{2\pi i \widetilde{h}\alpha}{3}} \int_{\mathbb{R} \mp \frac{\widetilde{h}i}{3}} \frac{e^{-\frac{\pi x^2}{kz} - 2\pi x\alpha - \frac{2\pi i u}{z}\left(x \pm \frac{\widetilde{h}i}{3}\right)}}{\cosh\left(\pi\left(x \pm \frac{\widetilde{h}i}{3}\right)\right)} dx \\ &= e^{-\frac{\pi \widetilde{h}^2}{9kz} \mp \frac{2\pi i \widetilde{h}\alpha}{3}} \int_{\mathbb{R}} \frac{e^{-\frac{\pi x^2}{kz} - 2\pi x\alpha - \frac{2\pi i u}{z}\left(x \pm \frac{\widetilde{h}i}{3}\right)}}{\cosh\left(\pi\left(x \pm \frac{\widetilde{h}i}{3}\right)\right)} dx, \end{aligned}$$

where for the last equality we used the Residue Theorem to shift the path back to $\mathbb{R}$ (noting that there are no poles in the shifted region). Differentiating then gives

$$H^{(\ell)}_{k,h,\alpha}(0; z) = e^{-\frac{\pi \widetilde{h}^2}{9kz} \mp \frac{2\pi i \widetilde{h}\alpha}{3}} \left(-\frac{2\pi i}{z}\right)^\ell \int_{\mathbb{R}} \left(x \pm \frac{i\widetilde{h}}{3}\right)^\ell \frac{e^{-\frac{\pi x^2}{kz} - 2\pi x\alpha}}{\cosh\left(\pi\left(x \pm \frac{\widetilde{h}i}{3}\right)\right)} dx.$$

Since $-\frac{1}{2} < \alpha < \frac{1}{2}$, we may bound

$$\frac{e^{-2\pi x\alpha}}{\cosh\left(\pi\left(x \pm \frac{\widetilde{h}i}{3}\right)\right)} \ll 1.$$

Thus

$$\left|H^{(\ell)}_{k,h,\alpha}(0; z)\right| \ll_\ell |z|^{-\ell}\, e^{-\frac{\pi \widetilde{h}^2}{9k} \mathrm{Re}\left(\frac{1}{z}\right)} \int_0^\infty x^\ell e^{-\frac{\pi x^2}{k} \mathrm{Re}\left(\frac{1}{z}\right)} dx.$$

A change of variables gives the bound stated in the lemma. $\square$

3.4. **Ranks.** In this section we prove an asymptotic result for the rank generating function. Recall (1.6).

Proposition 3.5. *Assuming the notation above, we have*

$$\begin{aligned} \mathcal{R}(u; q) = -i^{\frac{3}{2}} e^{\frac{\pi i}{12k}(h - [-h]_k)} \chi^{-1}\left(h, [-h]_k, k\right) e^{\frac{\pi}{12k}\left(\frac{1}{z} - z\right)} \sum_{r=0}^\infty \frac{(2\pi i u)^{2r}}{(2r)!} \\ \times \sum_{a+b+c=r} (3k)^a \kappa(a, b, c) z^{\frac{1}{2} - a - 2c} + \sum_{\ell=0}^\infty a_\ell(z) \frac{(2\pi i u)^\ell}{\ell!}. \end{aligned}$$

where the coefficients $a_\ell(z)$ *satisfy the asymptotic bounds* $a_\ell(z) \ll_\ell k^{\frac{1}{2}} |z|^{\frac{1}{2}-\ell}$.

In the following corollary we give the asymptotic expansion for each of the Taylor coefficients.

Corollary 3.6. *With the notation from above we have*

$$R_{2\ell}\left(e^{\frac{2\pi i}{k}(h+iz)}\right) = -i^{\frac{3}{2}} e^{\frac{\pi i}{12k}(h-[-h]_k)} \chi^{-1}\Big(h, [-h]_k, k\Big) e^{\frac{\pi}{12k}\left(\frac{1}{z}-z\right)} \sum_{a+b+c=\ell} (3k)^a \kappa(a,b,c) z^{\frac{1}{2}-a-2c} + a_{2\ell}(z),$$

where the coefficients $a_{2\ell}(z)$ *satisfy the asymptotic bounds* $a_{2\ell}(z) \ll_\ell k^{\frac{1}{2}} |z|^{\frac{1}{2}-2\ell}$.

Proof of Proposition 3.5. We consider first the case $3 \nmid k$. As in the case for the crank, we deduce the asymptotic behavior of ϑ and η from (2.5) and (2.3), yielding that the asymptotic of the modular term appearing in part (ii) of Proposition 2.6 is given by

$$-\frac{1}{3} i^{\frac{3}{2}} \chi^{-1}\left(h, [-h]_k, k\right) e^{\frac{\pi i}{12k}(h-[-h]_k)} \frac{1}{\sqrt{z}} e^{\frac{\pi}{12k}\left(\frac{1}{z}-z\right)} f_{3k}(u;z) + \sum_{\ell \geq 0} a_\ell(z) \frac{(2\pi i u)^\ell}{\ell!}, \tag{3.2}$$

with $a_\ell(z) \ll_\ell |z|^{\frac{1}{2}-\ell} e^{-\frac{\alpha}{k}\mathrm{Re}\left(\frac{1}{z}\right)}$ for some $\alpha > 0$ independent of k.

Using the asymptotic behavior of ϑ and observing that in the sum defining μ only the term $n = 0$ contributes to the asymptotic main term, we obtain that

$$\begin{aligned} \mu\left(\frac{iu}{z}, \mp\frac{k}{3}; \frac{1}{3k}\left([-h]_k + \frac{i}{z}\right)\right) &= \frac{e^{-\frac{\pi u}{z}}}{\vartheta\left(\mp\frac{k}{3}; \frac{1}{3k}\left([-h]_k + \frac{i}{z}\right)\right)} \sum_{n\in\mathbb{Z}} \frac{(-1)^n e^{\frac{\pi i}{3k}\left(n^2+n\right)\left([-h]_k + \frac{i}{z}\right)} e^{\mp\frac{2\pi i n k}{3}}}{1 - e^{-\frac{2\pi u}{z}} e^{\frac{2\pi i n}{3k}\left([-h]_k+\frac{i}{z}\right)}} \\ &= \pm\frac{(-1)^{k+1}\left(\frac{k}{3}\right) e^{-\frac{\pi i}{12k}\left([-h]_k+\frac{i}{z}\right)}}{2\sqrt{3}\sinh\left(\frac{\pi u}{z}\right)} + u^{-1}\sum_{\ell=0}^{\infty} a_\ell(z)\frac{(2\pi i u)^\ell}{\ell!}, \end{aligned}$$

with $a_\ell(z) \ll_\ell |z|^{1-\ell} e^{-\frac{\alpha}{k}\mathrm{Re}\left(\frac{1}{z}\right)}$ for some α independent of k. Therefore,

$$\begin{aligned} &\mp\frac{2\sin(\pi u)}{\sqrt{3z}} e^{-\frac{\pi z}{12k} - \frac{\pi i h}{4k} + \frac{3\pi k u^2}{z}} \xi(3h,k) \mu\left(\frac{iu}{z}, \mp\frac{k}{3}; \frac{1}{3k}\left([-h]_k + \frac{i}{z}\right)\right) \\ &= \frac{1}{3}(-1)^k \left(\frac{k}{3}\right) \xi(3h,k) e^{-\frac{\pi i h}{4k} - \frac{\pi i}{12k}[-h]_k} \frac{1}{\sqrt{z}} e^{\frac{\pi}{12k}\left(\frac{1}{z}-z\right)} f_{3k}(u;z) + \sum_{\ell\geq 0} \alpha_\ell(z) \frac{(2\pi i u)^\ell}{\ell!}, \end{aligned} \tag{3.3}$$

where $\alpha_\ell(z) \ll_\ell |z|^{\frac{1}{2}-\ell} e^{-\frac{\alpha}{k}\mathrm{Re}\left(\frac{1}{z}\right)}$ for some $\alpha > 0$ independent of k.

To give an asymptotic for the rank generating function it remains to consider the terms with the Mordell integrals H. We use Lemma 3.4 with $\alpha = \alpha^\pm(\ell, k)$. This implies that

$$\frac{\sin(\pi u)}{\sqrt{z}} e^{-\frac{\pi z}{12k} + \frac{3\pi k u^2}{z}} H\left(\frac{iu}{z} + \alpha^\pm(\ell,k); \frac{i}{3kz}\right) = \sum_{\ell\geq 0} a_\ell(z) \frac{(2\pi i u)^\ell}{\ell!}, \tag{3.4}$$

with $a_\ell(z) \ll_\ell |z|^{\frac{1}{2}-\ell}$.

Using (3.2), (3.3), and (3.4) with Proposition 2.6 we thus have

$$R\left(e^{2\pi iu}; e^{\frac{2\pi i}{k}(h+iz)}\right) = -\frac{i^{\frac{3}{2}}}{3} e^{\frac{\pi i}{12k}(h-[-h]_k)} \frac{f_{3k}(u;z)}{\sqrt{z}} e^{\frac{\pi}{12k}\left(\frac{1}{z}-z\right)} \chi^{-1}\left(h, [-h]_k, k\right)$$
$$\times \left(1 - 2(-1)^k \left(\frac{k}{3}\right) i^{-\frac{3}{2}} \xi(3h,k) \chi\left(h, [-h]_k, k\right) e^{-\frac{\pi i h}{3k}}\right)$$
$$+ \sum_{\ell \geq 0} a_\ell(z) \frac{(2\pi iu)^\ell}{\ell!},$$

with $a_\ell(z) \ll_\ell k^{\frac{1}{2}} |z|^{\frac{1}{2}-\ell}$. Applying Lemma 2.8 we obtain

(3.5)
$$R\left(e^{2\pi iu}; e^{\frac{2\pi i}{k}(h+iz)}\right) = -\, i^{\frac{3}{2}} \chi^{-1}\left(h, [-h]_k, k\right) e^{\frac{\pi i}{12k}(h-[-h]_k)} e^{\frac{\pi}{12k}\left(\frac{1}{z}-z\right)} \frac{f_{3k}(u;z)}{\sqrt{z}} + \sum_{\ell \geq 0} a_\ell(z) \frac{(2\pi iu)^\ell}{\ell!},$$

with $a_\ell(z) \ll_\ell k^{\frac{1}{2}} |z|^{\frac{1}{2}-\ell}$. Using Lemma 3.1 gives the claimed expansion for $3 \nmid k$.

Next we consider the case $3 \mid k$. Similarly as before we obtain that the contribution from the modular term of Proposition 2.6 is

$$-i^{\frac{3}{2}} z^{-\frac{1}{2}} e^{\frac{3\pi k u^2}{z}} \frac{\sin(\pi u)}{\sinh\left(\frac{3\pi u}{z}\right)} \chi^{-1}(h, [-h]_{3k}, k) e^{\frac{\pi i}{12k}(h-[-h]_{3k})} e^{\frac{\pi}{12k}\left(\frac{1}{z}-z\right)} + \sum_{\ell \geq 0} \alpha_\ell(z) \frac{(2\pi iu)^\ell}{\ell!},$$

with $a_\ell(z) \ll_\ell |z|^{\frac{1}{2}-\ell} e^{-\frac{\alpha}{k}\mathrm{Re}\left(\frac{1}{z}\right)}$ for some $\alpha > 0$ independent of k.

Next we turn to the asymptotics of the μ-terms from Proposition 2.6. Using that

$$\mu\left(\frac{3iu}{z}, \pm\widetilde{h}\tau; 3\tau\right) = \frac{e^{-\frac{3\pi u}{z}}}{\vartheta\left(\pm\widetilde{h}\tau; 3\tau\right)} \sum_{n \in \mathbb{Z}} \frac{(-1)^n e^{3\pi i(n^2+n)\tau \pm 2\pi i n \widetilde{h}\tau}}{1 - e^{-\frac{6\pi u}{z}} e^{6\pi i n\tau}}$$

gives that

$$\mu\left(\frac{3iu}{z}, \pm\frac{\widetilde{h}}{k}\left([-h]_{3k} + \frac{i}{z}\right); \frac{1}{k}\left([-h]_{3k} + \frac{i}{z}\right)\right) = \left(\frac{\pm\widetilde{h}}{3}\right) \frac{i e^{-\frac{\pi}{4kz} + \frac{\pi i [-h]_{3k}}{4k}}}{2\sinh\left(\frac{3\pi u}{z}\right)} + \sum_{\ell \geq 0} a_\ell(z) \frac{(2\pi iu)^\ell}{\ell!},$$

with $c_\ell(z) \ll_\ell |z|^{1-\ell} e^{-\frac{25\pi}{4k}\mathrm{Re}\left(\frac{1}{z}\right)}$.

Additionally we must consider the asymptotics of $H\left(\frac{3iu}{z} + \frac{\widetilde{h}i}{kz} + \alpha; \frac{3i}{kz}\right)$, with $\alpha = \alpha^{\pm}\left(\ell, \frac{k}{3}\right)$. We apply Lemma 3.4 to obtain

$$\frac{\sin(\pi u)}{\sqrt{z}} e^{-\frac{\pi z}{12k} + \frac{3\pi k u^2}{z} + \frac{\pi}{3kz}} H\left(\frac{3iu}{z} + \frac{\widetilde{h}i}{kz} + \alpha^{\pm}\left(\ell, \frac{k}{3}\right); \frac{3i}{kz}\right) = \sum_{\ell=0}^{\infty} a_\ell(z) \frac{(2\pi iu)^\ell}{\ell!},$$

with $a_\ell(z) \ll_\ell |z|^{\frac{1}{2}-\ell}$.

In total the asymptotic Taylor expansion for $R\left(e^{2\pi iu}; e^{\frac{2\pi i}{k}(h+iz)}\right)$ may be written as

$$-\frac{1}{\sqrt{z}}e^{\frac{\pi}{12k}\left(\frac{1}{z}-z\right)}e^{\frac{3\pi ku^2}{z}}\frac{\sin(\pi u)}{\sinh\left(\frac{3\pi u}{z}\right)}i^{\frac{3}{2}}e^{\frac{\pi i}{12k}(h-[-h]_{3k})}\chi^{-1}\left(h,[-h]_{3k},k\right)$$
$$\times\left(1+\sum_{\pm}\pm\left(\frac{\pm\widetilde{h}}{3}\right)e^{\mp\frac{2\pi\widetilde{h}u}{z}}\xi'\left(h,\frac{k}{3}\right)\chi\left(h,[-h]_{3k},k\right)i^{-\frac{1}{2}}e^{-\frac{\pi i}{3k}(h-[-h]_k)}\right)+\sum_{\ell\geq 0}a_\ell(z)\frac{(2\pi iu)^\ell}{\ell!},$$

with $a_\ell(z) \ll_\ell |z|^{\frac{1}{2}-\ell}$ Applying Lemma 2.8 when $3 \mid k$ we obtain that the asymptotic Taylor expansion for $R\left(e^{2\pi iu}; e^{\frac{2\pi i}{k}(h+iz)}\right)$ can be written as

$$-\frac{1}{\sqrt{z}}e^{\frac{\pi}{12k}\left(\frac{1}{z}-z\right)}f_{3k}(u;z)i^{\frac{3}{2}}e^{\frac{\pi i}{12k}(h-[-h]_{3k})}\chi^{-1}\left(h,[-h]_{3k},k\right)+\sum_{\ell\geq 0}a_\ell(z)\frac{(2\pi iu)^\ell}{\ell!},$$

with $a_\ell(z) \ll_\ell |z|^{\frac{1}{2}-\ell}$. Here we used that

$$\left(1+e^{-\frac{2\pi u}{z}}+e^{\frac{2\pi u}{z}}\right)\frac{1}{\frac{1}{2}\left(e^{\frac{3\pi u}{z}}-e^{-\frac{3\pi u}{z}}\right)}=\frac{1}{\sinh(\frac{\pi u}{z})}.$$

Therefore, in all cases we have established the asymptotic stated in Proposition 3.5. We note that all the occurring multipliers in the proposition are invariant under the choice of the inverse of h modulo k. □

4. Application of the Circle Method

Here we present a general Circle Method result that is applicable to both the rank and crank moment generating functions. Assume that

$$F_{r,\ell}\left(e^{\frac{2\pi i}{k}(h+iz)}\right)=\sum_n c_{r,\ell}(n)e^{\frac{2\pi in}{k}(h+iz)}$$

is a holomorphic function of z satisfying

(4.1)
$$F_{r,\ell}\left(e^{\frac{2\pi i}{k}(h+iz)}\right)=-i^{\frac{3}{2}}e^{\frac{\pi i}{12k}(h-[-h]_k)}\chi^{-1}\left(h,[-h]_k,k\right)e^{\frac{\pi}{12k}\left(\frac{1}{z}-z\right)}\sum_{a+b+c=\ell}(kr)^a\kappa(a,b,c)z^{\frac{1}{2}-a-2c}+E_{r,\ell,k}(z)$$

with $E_{r,\ell,k}(z) \ll_{r,\ell} k^{\frac{1}{2}}|z|^{\frac{1}{2}-2\ell}$ and $\kappa(a,b,c)$ defined as in (1.7).

Theorem 4.1. *With $F_{r,\ell}$ and $c_{r,\ell}$ as above we have*

$$c_{r,\ell}(n)=2\pi\sum_{k\leq\sqrt{n}}\frac{K_k(n)}{k}\sum_{a+b+c=\ell}(kr)^a\kappa(a,b,c)(24n-1)^{c+\frac{a}{2}-\frac{3}{4}}I_{\frac{3}{2}-2c-a}\left(\frac{\pi\sqrt{24n-1}}{6k}\right)+O\left(n^{2\ell-1+\epsilon}\right),$$

where $K_k(n)$ is the Kloosterman sum defined in (1.8) *and $\epsilon>0$.*

Proof. By Cauchy's Theorem for $n>0$ we have

$$c_{r,\ell}(n)=\frac{1}{2\pi i}\int_C\frac{F_{r,\ell}(q)}{q^{n+1}}dq,$$

with C an arbitrary path inside the unit circle that encloses 0 with a counterclockwise orientation. Choosing the circle with radius $e^{-\frac{2\pi}{n}}$ and parameterized by $q = e^{-\frac{2\pi}{n}+2\pi it}$ with $0 \leq t \leq 1$ yields

$$c_{r,\ell}(n) = \int_0^1 F_{r,\ell}\left(e^{-\frac{2\pi}{n}+2\pi it}\right) e^{2\pi - 2\pi int} dt.$$

Define

$$\vartheta'_{h,k} := \frac{1}{k(k_1+k)} \qquad \text{and} \qquad \vartheta''_{h,k} := \frac{1}{k(k_2+k)},$$

where $\frac{h_1}{k_1} < \frac{h}{k} < \frac{h_2}{k_2}$ are adjacent Farey fractions in the Farey sequence of order $N := \lfloor n^{\frac{1}{2}} \rfloor$. Recall that $\frac{1}{k+k_j} \leq \frac{1}{N+1}$ for $j = 1, 2$. Next, decompose the path of integration into paths along the Farey arcs $-\vartheta'_{h,k} \leq \Phi \leq \vartheta''_{h,k}$, where $\Phi := t - \frac{h}{k}$ and $0 \leq h \leq k \leq N$ with $(h,k) = 1$. Hence

$$c_{r,\ell}(n) = \sum_{h,k} e^{-\frac{2\pi ihn}{k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} F_{r,\ell}\left(e^{\frac{2\pi i}{k}(h+iz)}\right) e^{\frac{2\pi nz}{k}} d\Phi,$$

where $z = \frac{k}{n} - k\Phi i$. We apply (4.1) and estimate the error as follows. Using $\mathrm{Re}\left(\frac{1}{z}\right) > \frac{k}{2}$, $\mathrm{Re}\,(z) = \frac{k}{n}$, $|z| \geq \frac{k}{n}$ and $\vartheta_{h,k}, \vartheta''_{h,k} < \frac{1}{k(N+1)}$, we can bound the terms arising from $E_{r,\ell,k}$ by

$$\sum_{h,k} \int_{-\vartheta_{h,k}}^{\vartheta''_{h,k}} |E_{r,\ell,k}(z)|\, d\Phi \ll \frac{1}{\sqrt{n}} \sum_{k \leq N} k^{\frac{1}{2}} \left(\frac{k}{n}\right)^{\frac{1}{2}-2\ell} \ll n^{2\ell-1}.$$

Thus we obtain

$$c_{r,\ell}(n) = -i^{\frac{3}{2}} \sum_{h,k} e^{-\frac{2\pi ihn}{k}} e^{\frac{\pi i}{12k}(h-[-h]_k)} \chi^{-1}\Big(h, [-h]_k, k\Big) \sum_{a+b+c=\ell} (kr)^a \kappa(a,b,c)$$
$$\times \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} e^{\frac{\pi}{12k}\left((24n-1)z+\frac{1}{z}\right)} z^{\frac{1}{2}-a-2\ell} d\Phi + O\Big(n^{2\ell-1}\Big).$$

The theorem now follows from the following integral evaluation (see [23] for details)

$$\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} e^{\frac{\pi}{12k}\left((24n-1)z+\frac{1}{z}\right)} z^{\frac{1}{2}-j} d\Phi = \frac{2\pi}{k}\,(24n-1)^{\frac{j}{2}-\frac{3}{4}}\, I_{\frac{3}{2}-j}\left(\frac{\pi\sqrt{24n-1}}{6k}\right) + O\Big(n^{2\ell-1}\Big)$$

and the remark after (2.4). □

The proofs of Theorem 1.1 and Corollary 1.2 follow easily.

Proof of Theorem 1.1. Corollaries 3.3 and 3.6 imply that the rank and crank moment generating functions satisfy the assumptions necessary to apply Theorem 4.1. Therefore, Theorem 4.1 implies the desired result. □

The symmetrized crank and rank moments studied by Garvan [22] and Andrews [1] are defined by

$$\eta_k(n) := \sum_{m=-n}^{n} \binom{m + \lfloor \frac{k-1}{2} \rfloor}{k} \mathcal{M}(m,n), \tag{4.2}$$

$$\nu_k(n) := \sum_{m=-n}^{n} \binom{m + \lfloor \frac{k-1}{2} \rfloor}{k} \mathcal{N}(m,n). \tag{4.3}$$

Proof of Corollary 1.2. The proof of the first formula follows from (4.2) of [22] which gives

$$\eta_{2\ell}(n) = \frac{1}{(2\ell)!}\sum_{m=-n}^{n} g_\ell(m)\mathcal{N}(m,n)$$

where $g_\ell(x) = \prod_{j=0}^{\ell-1}(x^2 - j^2)$ and from Theorem 1.1. The second formula is analogous. □

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