PARTITION IDENTITIES AND A THEOREM OF ZAGIER

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Abstract. In the study of partition theory and q-series, identities that relate series to infinite products are of great interest (such as the famous Rogers-Ramanujan identities). Using a recent result of Zagier, we obtain an infinite family of such identities that is indexed by the positive integers. For example, if $m = 1$, then we obtain the classical Eisenstein series identity

$$\sum_{\lambda \geq 1 \text{ odd}} \frac{(-1)^{(\lambda-1)/2} q^{\lambda}}{(1-q^{2\lambda})} = q \prod_{n=1}^{\infty} \frac{(1-q^{8n})^4}{(1-q^{4n})^2}.$$

If $m = 2$ and $\left(\frac{\cdot}{3}\right)$ denotes the usual Legendre symbol modulo 3, then we obtain

$$\sum_{\lambda \geq 1} \frac{\left(\frac{\lambda}{3}\right) q^{\lambda}}{(1-q^{2\lambda})} = q \prod_{n=1}^{\infty} \frac{(1-q^n)(1-q^{6n})^6}{(1-q^{2n})^2(1-q^{3n})^3}.$$

We describe some of the partition theoretic consequences of these identities. In particular, we find simple formulas that solve the well-known problem of counting the number of representations of an integer as a sum of an arbitrary number of triangular numbers.

1. Introduction and Statement of Results

Recall the following identity of Euler:

$$\text{(Euler)} \qquad \prod_{n=1}^{\infty} (1-q^n) = \sum_{k=-\infty}^{\infty} (-1)^k q^{(3k^2+k)/2}.$$

Note that the left hand side of Euler's identity is related to partitions of integers into distinct parts. More precisely, each partition of n into an odd number of distinct parts adds -1 to the coefficient of q^n and each partition of n into an even number of distinct parts adds $+1$.

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Therefore, this identity, known as Euler's pentagonal number theorem, shows us that the number of partitions of n into an odd number of distinct parts is equal to the number of partitions of n into an even number of distinct parts except when n is a pentagonal number, that is, a number of the form $(3k^2+k)/2$.

There are many other identities of a similar form equating an infinite q-series product to an infinite sum. Perhaps the most famous are the Rogers-Ramanujan identities, one of which follows:

$$\text{(Rogers-Ramanujan)} \qquad \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})} = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)}.$$

A combinatorial interpretation of this identity establishes that the number of partitions of an integer n into parts that are congruent to $2, 3 \pmod 5$ equals the number of partitions of n in which any two summands differ by at least 2 and all summands exceed 1.

The purpose of this paper is to use a master theorem of Zagier [Z] previously conjectured by Kac and Wakimoto [K-W] to prove a natural infinite family of analogous simple q-series identities that relate infinite products to generating functions for certain restricted partition functions. Through a combinatorial examination of these identities, we obtain closed explicit formulas for the number of representations of integers as a sum of an arbitrary number of triangular numbers. We note here that the general problem of finding formulas for the number of representations of n as a sum of squares and triangular numbers has seen great advances in the recent works of Milne, Ono, and Zagier (see [M], [O], [Z]).

Before we state the main theorems we must offer some notation. Throughout, if m is a positive integer, then let $s(m)$ denote the integer

$$s(m) := \left[\frac{m+1}{2}\right]. \tag{1.1}$$

For convenience, we define the following set of vectors which will determine the summands in our partition functions.

Definition 1.1. *If m is a positive integer, then let $S(m)$ denote the set of integral vectors $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{s(m)})$ for which the following hold:*

- *(i) $\lambda_1 > 0$ and if $j > i$, then $\lambda_i > \lambda_j > 0$.*
- *(ii) For every i we have $\lambda_i \equiv m \pmod 2$.*
- *(iii) For every i we have $\lambda_i \not\equiv 0 \pmod{2m+2}$.*
- *(iv) For every $i \neq j$ we have $\lambda_i \not\equiv \pm\lambda_j \pmod{2m+2}$.*

Further, define subsets $S_{\pm}(m)$ of $S(m)$ in the following manner:

$$S_+(m) := \{\Lambda \in S(m) \ : \ \text{the number of } \lambda_i \pmod{2m+2} > m+1 \text{ is even}\}, \tag{1.2}$$

$$S_-(m) := \{\Lambda \in S(m) \ : \ \text{the number of } \lambda_i \pmod{2m+2} > m+1 \text{ is odd}\}. \tag{1.3}$$

Now we can state our primary results, which will be proved in Section 2:

Theorem 1.1. *If $m \geq 1$ is odd, then*

$$\sum_{\Lambda \in S_+(m)} \frac{q^{(\lambda_1+\cdots+\lambda_{s(m)})/2}}{(1-q^{\lambda_1})\cdots(1-q^{\lambda_{s(m)}})} - \sum_{\Lambda \in S_-(m)} \frac{q^{(\lambda_1+\cdots+\lambda_{s(m)})/2}}{(1-q^{\lambda_1})\cdots(1-q^{\lambda_{s(m)}})}$$

$$= q^{(m+1)^2/8} \prod_{n=1}^{\infty} \frac{(1-q^{2(m+1)n})^{2m+2}}{(1-q^{(m+1)n})^{m+1}}.$$

Theorem 1.2. *If $m \geq 1$ is even, then*

$$\sum_{\Lambda \in S_+(m)} \frac{q^{(\lambda_1+\cdots+\lambda_{s(m)})/2}}{(1-q^{\lambda_1})\cdots(1-q^{\lambda_{s(m)}})} - \sum_{\Lambda \in S_-(m)} \frac{q^{(\lambda_1+\cdots+\lambda_{s(m)})/2}}{(1-q^{\lambda_1})\cdots(1-q^{\lambda_{s(m)}})}$$

$$= q^{(m^2+2m)/8} \prod_{n=1}^{\infty} \frac{(1-q^n)}{(1-q^{2n})^2} \frac{(1-q^{2(m+1)n})^{2m+2}}{(1-q^{(m+1)n})^{m+1}}.$$

Before stating some partition theoretic interpretations of these identities, we must offer additional definitions.

Definition 1.2. *If m is a positive integer, then let $P_{\pm}(n,m)$ denote the number of partitions of n of the form*

$$n = \sum_{i=1}^{s(m)} (2n_i+1)\lambda_i,$$

where $\Lambda = (\lambda_1, \ldots, \lambda_{s(m)}) \in S_{\pm}(m)$ and each $n_i \geq 0$.

In simple terms, the function $P_{\pm}(n,m)$ counts the number of partitions of n into parts that are elements of a vector $\Lambda \in S_{\pm}(m)$, with odd multiplicity for each part. Clearly, for a fixed m,

$$\sum_{\Lambda \in S_{\pm}(m)} \frac{q^{(\lambda_1+\cdots+\lambda_{s(m)})}}{(1-q^{2\lambda_1})\cdots(1-q^{2\lambda_{s(m)}})} = \sum_{n=1}^{\infty} P_{\pm}(n,m)q^n, \tag{1.4}$$

so the left side of our theorems is the difference of the generating functions for $P_{\pm}(n,m)$.

Moreover, the infinite products in our theorems are related to the generating functions for the number of representations of an integer as a sum of triangular numbers, i.e. numbers of the form $(k^2+k)/2$ with $k \geq 0$. If k is a positive integer, then let $T(n,k)$ denote the number of representations of n as a sum of k triangular numbers. A well known identity due to Jacobi implies that

$$\sum_{n=0}^{\infty} T(n,k)q^n = \left(\prod_{n=1}^{\infty} \frac{(1-q^{2n})^2}{(1-q^n)} \right)^k. \tag{1.5}$$

In view of (1.4) and (1.5) it is simple to verify the following corollaries of Theorems 1.1 and 1.2, which give partition theoretic formulas for $T(n,k)$.

Corollary 1.3. *If $k \geq 2$ is even, then*

$$T(n,k) = P_+\left(2kn + \frac{k^2}{4}, k-1\right) - P_-\left(2kn + \frac{k^2}{4}, k-1\right).$$

Corollary 1.4. *If $k \geq 1$ is odd, then*

$$T(n,k) = \sum_{j=0}^{\infty}\left(P_+\left(2kn + \frac{k^2-1}{4} - j^2 - j, k-1\right) - P_-\left(2kn + \frac{k^2-1}{4} - j^2 - j, k-1\right)\right).$$

Remark. Observe that Corollaries 1.3 and 1.4 completely characterize the number of representations of every integer n as a sum of an arbitrary number of triangular numbers in terms of the partition functions $P_\pm(n,k)$. One should compare these results with those appearing in [M] where explicit formulas of a different type are obtained for $T(n,k)$ for those k of the form $4s^2$ and $4s^2+4s$. It would be very interesting to obtain a combinatorial proof of these results.

Along with (1.4), Theorems 1.1 and 1.2 immediately imply the following corollaries.

Corollary 1.5. *If $k \geq 2$ is even, then for every non-negative integer n we have*

$$P_+(n,k-1) \geq P_-(n,k-1).$$

Furthermore, if $n \not\equiv k^2/4 \pmod{2k}$, then

$$P_+(n,k-1) = P_-(n,k-1).$$

Corollary 1.6. *If $k \geq 2$ is even and n is a positive odd integer, then*

$$P_+(n,k) = P_-(n,k).$$

Remark. If $k \geq 4$ is even and $n \equiv k^2/4 \pmod{2k}$, then $P_+(n,k-1) > P_-(n,k-1)$ by Gauss' Eureka Theorem (i.e. $T(n,3) > 0$ for all n). When $k=2$, an easy analysis shows that $P_+(n,1) = P_-(n,1)$ for a set of positive integers n with arithmetic density one.

One may also make the observation that the right hand side of the identities of Theorems 1.1 and 1.2 are quotients of powers of Dedekind eta-functions. In Section 3 we use well known results on such eta-products to show that the generating function for $P_+(n,m) - P_-(n,m)$ is a holomorphic integer weight modular form. Then, using a powerful result of Serre, we obtain the following corollary:

Corollary 1.7. *If k is a positive integer and M is any integer, then*

$$P_+(n,k) \equiv P_-(n,k) \pmod{M}$$

for a set of positive integers n with arithmetic density 1.

Notice that Corollary 1.7 is a weak analog of Euler's pentagonal number theorem mentioned above.

2. Proof of Theorems 1.1 and 1.2

Our proof of Theorems 1.1 and 1.2 relies on the Kac-Wakimoto Conjecture, which was proved by Zagier. The functions

$$\Theta_0(x) := \prod_{n=1}^{\infty}(1-q^{2n})(1-q^{2n-1}x)(1-q^{2n-1}x^{-1})$$
$$\Theta_1(x) := \prod_{n=1}^{\infty}(1-q^{2n})(1-q^{2n-2}x)(1-q^{2n}x^{-1})$$

appear in the result, where the complex number q is fixed throughout this section. To prove our theorems, we use the following form of the Conjecture, which appears as an intermediate step in Zagier's proof (see [Z]):

Theorem 2.1. *(Zagier) If $m \in \mathbb{Z}^+, s(m) = \left[\frac{m+1}{2}\right]$ and $x_1, x_2, \ldots, x_{m+1} \in \mathbb{C}^*$ are distinct complex numbers such that $|q| < |x_i/x_j| < |q|^{-1}$ for all $1 \leq i, j \leq m+1$, then*

$$\sum_{\substack{\lambda_1 > \lambda_2 > \cdots > \lambda_{s(m)} > 0 \\ \lambda_i \equiv m \pmod 2}} \frac{q^{(\lambda_1+\cdots+\lambda_{s(m)})/2}}{(1-q^{\lambda_1})(1-q^{\lambda_2})\ldots(1-q^{\lambda_{s(m)}})} \times (\mathrm{Alt})_{m+1}\left(\prod_{i=1}^{s(m)}\left(\frac{x_i}{x_{m+2-i}}\right)^{\lambda_i/2}\right)$$
$$= \left(q^{1/8}\prod_{n=1}^{\infty}\frac{(1-q^{2n})^2}{(1-q^n)}\right)^{2s(m)} \prod_{1\leq i<j\leq m+1} F(x_j/x_i), \tag{2.1}$$

where

$$F(x) := q^{1/4}x^{-1/2}\frac{\Theta_1(x)}{\Theta_0(x)}$$

and Alt_{m+1} denotes the alternating sum over all permutations of $x_1, \ldots, x_{m+1}$.

We derive Theorems 1.1 and 1.2 directly from Theorem 2.1 by first substituting roots of unity for the x_i, and then using combinatorial arguments to relate the complicated alternating sum to known values.

Proof of Theorems 1.1 and 1.2. Our first task is to simplify the right side of (2.1). Let us rewrite the functions Θ_0 and Θ_1 as follows:

$$\Theta_0(x) = \prod_{n=1}^{\infty}(1-q^{2n-1})(1-q^{2n-1}x)(1-q^{2n-1}x^{-1}) \cdot \prod_{n=1}^{\infty}\frac{(1-q^{2n})}{(1-q^{2n-1})}$$
$$= \prod_{n=1}^{\infty}(1-q^{2n-1})(1-q^{2n-1}x)(1-q^{2n-1}x^{-1}) \cdot \prod_{n=1}^{\infty}\frac{(1-q^{2n})^2}{(1-q^{n})}, \tag{2.2}$$

$$\Theta_1(x) = (1-x)\prod_{n=1}^{\infty}(1-q^{2n})(1-q^{2n}x)(1-q^{2n}x^{-1}). \tag{2.3}$$

Note that for all $i < j$, the term $(x_j/x_i)^{-1/2}$ is a factor of $F(x_j/x_i)$. Since we will be substituting complex values for the x_i, the branching of the square root function necessitates a careful treatment of these terms. For now, move all such terms to the left side of the equation.

We must also check the exponents in Θ_1 and Θ_0:

$$\prod_{1\le i<j\le m+1} \Theta_1(x_j/x_i) = \prod_{1\le i<j\le m+1} \left((1-x_j/x_i) \prod_{n=1}^{\infty} (1-q^{2n})(1-q^{2n}(x_j/x_i))(1-q^{2n}(x_j/x_i)^{-1}) \right). \tag{2.4}$$

Each x_i appears with integral exponents, and $\Theta_0(x_j/x_i)$ has a similar form. Thus we can make the substitution $x_i = \zeta^{i-1}, 1 \le i \le m+1$, where ζ is a primitive $(m+1)$-st root of unity. The value of m is fixed throughout this proof, so this notation will cause no confusion. The fraction (x_j/x_i) now reduces to ζ^{j-i}, and since the set $\{1, \zeta, \zeta^2, \ldots, \zeta^m\}$ contains all $(m+1)$-st roots of unity,

$$(1-q^{2n})(1-q^{2n}\zeta)\cdots(1-q^{2n}\zeta^m) = (1-q^{2n(m+1)}).$$

The product (2.4) contains $m+1$ copies of the term $\prod_{n=1}^{\infty}(1-q^{2n}\zeta^k)$ for all $1 \le k \le m$. Thus we have

$$\begin{aligned}
\prod_{1\le i<j\le m+1} & \Theta_1(\zeta^{j-i}) \\
&= \prod_{1\le i<j\le m+1} (1-\zeta^{j-i}) \prod_{n=1}^{\infty} (1-q^{2n})^{(m+1)m/2}(1-q^{2n}\zeta)^{m+1}\cdots(1-q^{2n}\zeta^m)^{m+1} \\
&= \prod_{k=1}^{m}(1-\zeta^k)^{m+1-k} \prod_{n=1}^{\infty}(1-q^{2n})^{(m+1)(m-2)/2}(1-q^{2n(m+1)})^{m+1}.
\end{aligned}$$

The same substitution for x_i yields

$$\begin{aligned}
\prod_{1\le i<j\le m+1} & \Theta_0(\zeta^{j-i}) \\
&= \prod_{n=1}^{\infty}(1-q^{2n-1})^{(m+1)(m-2)/2}(1-q^{(2n-1)(m+1)})^{m+1} \prod_{n=1}^{\infty} \frac{(1-q^{2n})^{(m+1)m}}{(1-q^n)^{(m+1)m/2}}.
\end{aligned}$$

After these simplifications (including the transfer of the fractional powers of x_i to the left side), the right side of (2.1) becomes

$$\begin{aligned}
& q^{(m^2+m+2s(m))/8} \prod_{k=1}^{m}(1-\zeta^k)^{m+1-k} \\
& \quad \times \prod_{n=1}^{\infty} \frac{(1-q^n)^{(m+1)m/2-2s(m)}}{(1-q^{2n})^{(m+1)(m+2)/2-4s(m)}(1-q^{2n-1})^{(m+1)(m-2)/2}} \prod_{n=1}^{\infty} \frac{(1-q^{(m+1)2n})^{2m+2}}{(1-q^{(m+1)n})^{m+1}}.
\end{aligned}$$

If m is odd, then $2s(m) = m + 1$, so the above product is equal to

$$(2.5) \quad q^{(m+1)^2/8} \prod_{k=1}^{m} (1-\zeta^k)^{m+1-k} \prod_{n=1}^{\infty} \left(\frac{(1-q^n)}{(1-q^{2n})(1-q^{2n-1})} \right)^{(m+1)(m-2)/2} \times \prod_{n=1}^{\infty} \frac{(1-q^{(m+1)2n})^{2m+2}}{(1-q^{(m+1)n})^{m+1}}$$
$$= \quad q^{(m+1)^2/8} \prod_{k=1}^{m} (1-\zeta^k)^{m+1-k} \prod_{n=1}^{\infty} \frac{(1-q^{(m+1)2n})^{2m+2}}{(1-q^{(m+1)n})^{m+1}},$$

which is nearly the form seen in the right side of Theorem 1.1. And when m is even, $2s(m) = m$, so the product is equal to

$$(2.6) \quad q^{(m^2+2m)/8} \prod_{k=1}^{m} (1-\zeta^k)^{m+1-k} \prod_{n=1}^{\infty} \frac{(1-q^n)}{(1-q^{2n})^2} \left(\frac{(1-q^n)}{(1-q^{2n})(1-q^{2n-1})} \right)^{(m+1)(m-2)/2} \times \prod_{n=1}^{\infty} \frac{(1-q^{(m+1)2n})^{2m+2}}{(1-q^{(m+1)n})^{m+1}}$$
$$= \quad q^{(m^2+2m)/8} \prod_{k=1}^{m} (1-\zeta^k)^{m+1-k} \prod_{n=1}^{\infty} \frac{(1-q^n)}{(1-q^{2n})^2} \frac{(1-q^{(m+1)2n})^{2m+2}}{(1-q^{(m+1)n})^{m+1}}.$$

This resembles the right side of Theorem 1.2.

If (2.5) and (2.6) are expanded into an infinite sum, then the first nonzero term with positive exponent in q has the form $cq^{(m+1)^2/8}$ when m is odd, and $cq^{(m^2+2m)/8}$ when m is even, where

$$c := \prod_{k=1}^{m} (1-\zeta^k)^{m+1-k}.$$

Now we simplify the left side of (2.1). We must include the product of the transferred radical terms:

$$X := \prod_{1 \le i < j \le m+1} (x_j/x_i)^{1/2} = \prod_{k=1}^{m+1} (x_k)^{(2k-m-2)/2}.$$

Thus we have the following:
(2.7)

$$\sum_{\substack{\lambda_1 > \lambda_2 > \cdots > \lambda_{s(m)} > 0 \\ \lambda_i \equiv m \pmod 2}} \frac{q^{(\lambda_1 + \cdots + \lambda_{s(m)})/2}}{(1-q^{\lambda_1})(1-q^{\lambda_2}) \ldots (1-q^{\lambda_{s(m)}})} \times (\mathrm{Alt})_{m+1} \left(\prod_{i=1}^{s(m)} \left(\frac{x_i}{x_{m+2-i}} \right)^{\lambda_i/2} \right) X.$$

Before we substitute roots of unity for the x_i, we check that all of their exponents are integral. If m is even, then each λ_i is even, so $\lambda_i/2$ is integral, and each exponent in X is also integral.

If m is odd, then each λ_i is also odd, but so is $(2k-m-2)$, which is the numerator of the exponent for x_k in X. Since m is odd, every one of the $m+1$ variables appears in each term of the alternating sum, so the total exponent for each x_k is an integer. Thus we may set $x_i = \zeta^{i-1}$. The value of the alternating sum now depends only on Λ, so we define

$$(2.8)\quad A(\Lambda) := (\mathrm{Alt})_{m+1}\left(\prod_{i=1}^{s(m)}\left(\frac{x_i}{x_{m+2-i}}\right)^{\lambda_i/2}\right) = \sum_{\sigma\in S_{m+1}}\mathrm{sgn}(\sigma)\prod_{i=1}^{s(m)}\left(\frac{x_{\sigma(i)}}{x_{\sigma(m+2-i)}}\right)^{\lambda_i/2}$$
$$= \sum_{\sigma\in S_{m+1}}\mathrm{sgn}(\sigma)\prod_{i=1}^{s(m)}\left(\frac{\zeta^{\sigma(i)-1}}{\zeta^{\sigma(m+2-i)-1}}\right)^{\lambda_i/2}.$$

We also define $\bar{X}$ to be X evaluated with this same substitution, i.e.

$$\bar{X} := \prod_{k=1}^{m+1}(\zeta^{k-1})^{(2k-m-2)/2}.$$

For all but the simplest cases, the alternating sum is too unwieldy to calculate directly. Instead we evaluate it by comparing the series expansion of the left and right sides of (2.1). The exponent of q for an arbitrary term on the left side is half of the sum of the components of some vector Λ, and hence the term of minimum degree in q corresponds to the unique vector of minimum sum, namely,

$$\Lambda' = (\lambda'_1,\ldots,\lambda'_{s(m)}) := \begin{cases}(m, m-2,\ldots,2) & \text{when } m \text{ is even,}\\ (m, m-2,\ldots,1) & \text{when } m \text{ is odd.}\end{cases}$$

The minimum exponent is $(\lambda'_1+\cdots+\lambda'_{s(m)})/2$, which evaluates to $\frac{1}{2}(\frac{m+1}{2})^2 = \frac{(m+1)^2}{8}$ when m is odd and $\frac{1}{2}(2(\frac{m}{2}(\frac{m}{2}+1)/2)) = \frac{m^2+2m}{8}$ when m is even. These values are precisely the exponents of least degree on the right hand side of (2.5) and (2.6). The series expansion of the left side must correspond to that of the right side, so we have the following identity

$$A(\Lambda')\bar{X} = c = \prod_{k=1}^{m}(1-\zeta^k)^{m+1-k}.$$

We will now use (2.8) to show that $A(\Lambda) = -A(\Lambda)$ (and is hence zero) for all $\Lambda \notin S(m)$, and that $A(\Lambda) = \pm A(\Lambda')$ for $\Lambda \in S(m)$. Note that we need only consider vectors modulo $(2m+2)$, for $(\zeta^{i-1}/\zeta^{m+1-i}) = \zeta^{2i-m-1}$ is an $(m+1)$-st root of unity.

Consider the case where $\lambda_j \equiv 0 \pmod{2m+2}$ for some $1 \le j \le s(m)$. Then

$$(\zeta^{\sigma(j)-1}/\zeta^{\sigma(m+2-j)-1})^{\lambda_j/2} = 1$$

for any permutation $\sigma \in S_{m+1}$, so this term can be ignored in the alternating sum. We now define the transposition $\tau = (j \;\; m+2-j)$, and note that $\sigma\tau(i) = \sigma(i)$ for all $i \neq j, m+2-j$, but $\operatorname{sgn}(\sigma\tau) = -\operatorname{sgn}(\sigma)$. Thus we can negate (2.8) by inserting τ:

$$A(\Lambda) = \sum_{\sigma \in S_{m+1}} \operatorname{sgn}(\sigma) \prod_{1 \leq i \leq s(m),\, i \neq j} \left(\frac{\zeta^{\sigma(i)-1}}{\zeta^{\sigma(m+2-i)-1}} \right)^{\lambda_i/2}$$
$$= \sum_{\sigma \in S_{m+1}} (-1) \operatorname{sgn}(\sigma\tau) \prod_{1 \leq i \leq s(m),\, i \neq j} \left(\frac{\zeta^{\sigma\tau(i)-1}}{\zeta^{\sigma\tau(m+2-i)-1}} \right)^{\lambda_i/2}$$
$$= - \sum_{\sigma\tau \in S_{m+1}} \operatorname{sgn}(\sigma\tau) \prod_{1 \leq i \leq s(m),\, i \neq j} \left(\frac{\zeta^{\sigma\tau(i)-1}}{\zeta^{\sigma\tau(m+2-i)-1}} \right)^{\lambda_i/2} = -A(\Lambda).$$

The third equality follows since the summation index is over the group S_{m+1}, and thus the mapping that sends σ to $\sigma\tau$ is a bijection. Thus $A(\Lambda) = 0$ for vectors of this form.

Now consider the case where $\lambda_j \equiv \lambda_k \pmod{2m+2}$ for some $1 \leq j, k \leq s(m)$. Let $\tau = (j \;\; k)$. The j-th and k-th terms can then be combined under their common exponent, and for any σ,

$$\left(\frac{\zeta^{\sigma(j)+\sigma(k)-2}}{\zeta^{\sigma(m+2-j)+\sigma(m+2-k)-2}} \right)^{\lambda_j/2} = \left(\frac{\zeta^{\sigma\tau(k)+\sigma\tau(j)-2}}{\zeta^{\sigma\tau(m+2-j)+\sigma\tau(m+2-k)-2}} \right)^{\lambda_j/2}.$$

The denominators are equal because $\sigma\tau(i) = \sigma(i)$ for all $i \neq j, k$. Again we insert τ into (2.8):

$$A(\Lambda) = \sum_{\sigma \in S_{m+1}} \operatorname{sgn}(\sigma) \left(\frac{\zeta^{\sigma(j)+\sigma(k)-2}}{\zeta^{\sigma(m+2-j)+\sigma(m+2-k)-2}} \right)^{\lambda_j/2} \prod_{\substack{1 \leq i \leq s(m) \\ i \neq j,k}} \left(\frac{\zeta^{\sigma(i)-1}}{\zeta^{\sigma(m+2-i)-1}} \right)^{\lambda_i/2}$$
$$= \sum_{\sigma \in S_{m+1}} (-1) \operatorname{sgn}(\sigma\tau) \left(\frac{\zeta^{\sigma\tau(k)+\sigma\tau(j)-2}}{\zeta^{\sigma\tau(m+2-j)+\sigma\tau(m+2-k)-2}} \right)^{\lambda_j/2} \prod_{\substack{1 \leq i \leq s(m) \\ i \neq j,k}} \left(\frac{\zeta^{\sigma\tau(i)-1}}{\zeta^{\sigma\tau(m+2-i)-1}} \right)^{\lambda_i/2}$$
$$= - \sum_{\sigma\tau \in S_{m+1}} \operatorname{sgn}(\sigma\tau) \prod_{1 \leq i \leq s(m)} \left(\frac{\zeta^{\sigma\tau(i)-1}}{\zeta^{\sigma\tau(m+2-i)-1}} \right)^{\lambda_i/2} = -A(\Lambda).$$

Once again the change in the summation index is valid, and thus $A(\Lambda) = 0$ for Λ of this form.

Next, suppose that $\lambda_j \equiv -\lambda_k \pmod{2m+2}$ for some $1 \leq j, k \leq s(m)$. This case reduces to the previous one after an easy manipulation. Let $\tau = (j \;\; m+2-j)$, so for any σ,

$$\left(\frac{\zeta^{\sigma(j)-1}}{\zeta^{\sigma(m+2-j)-1}} \right)^{\lambda_j/2} = \left(\frac{\zeta^{\sigma(m+2-j)-1}}{\zeta^{\sigma(j)-1}} \right)^{-\lambda_j/2} = \left(\frac{\zeta^{\sigma\tau(j)-1}}{\zeta^{\sigma\tau(m+2-j)-1}} \right)^{\lambda_k/2}.$$

Define a new vector $\Gamma = (\gamma_1, \ldots, \gamma_{s(m)})$ by $\gamma_j = -\lambda_j$ and $\gamma_i = \lambda_i$ for $i \neq j$, so that $\gamma_j \equiv \gamma_k \pmod{2m+2}$. We now show that $A(\Lambda) = -A(\Gamma)$, which is zero by the previous case.

$$\begin{aligned}
A(\Lambda) &= \sum_{\sigma \in S_{m+1}} \operatorname{sgn}(\sigma) \left(\frac{\zeta^{\sigma(j)-1}}{\zeta^{\sigma(m+2-j)-1}} \right)^{\lambda_j/2} \prod_{\substack{1 \leq i \leq s(m) \\ i \neq j}} \left(\frac{\zeta^{\sigma(i)-1}}{\zeta^{\sigma(m+2-i)-1}} \right)^{\lambda_i/2} \\
&= \sum_{\sigma \in S_{m+1}} (-\operatorname{sgn}(\sigma\tau)) \left(\frac{\zeta^{\sigma\tau(j)-1}}{\zeta^{\sigma\tau(m+2-j)-1}} \right)^{\gamma_j/2} \prod_{\substack{1 \leq i \leq s(m) \\ i \neq j}} \left(\frac{\zeta^{\sigma\tau(i)-1}}{\zeta^{\sigma\tau(m+2-i)-1}} \right)^{\gamma_i/2} \\
&= -\sum_{\sigma\tau \in S_{m+1}} \operatorname{sgn}(\sigma\tau) \prod_{i=1}^{s(m)} \left(\frac{\zeta^{\sigma\tau(i)-1}}{\zeta^{\sigma\tau(m+2-i)-1}} \right)^{\gamma_i/2} = -A(\Gamma) = 0.
\end{aligned}$$

The only remaining vectors are those for which there is at most one λ_i in each pair of residue classes, k and $2m+2-k$ modulo $(2m+2)$. In fact, the following arguments show that there must be exactly one λ_i in each such pair. Each λ_i has the same parity as m, so there are at most $m+1$ possible values modulo $(2m+2)$. We have also disallowed any pair of additive inverses modulo $(2m+2)$, so the number of possible values is halved again, leaving $\left[\frac{m+1}{2}\right] = s(m)$ choices. We already know that the vector Λ' has $s(m)$ distinct terms, and because of the symmetry of the alternating sum, the value of $A(\Lambda)$ does not depend on the order of the vector components. Therefore, after reordering, the components of Λ must satisfy $\lambda_i \equiv \pm\lambda'_i \pmod{2m+2}$. We also know that $1 \leq \lambda'_i < m+1$ for all i, so $m+2 \leq -\lambda'_i \pmod{2m+2} \leq 2m+2$. Thus if Λ has exactly r terms in the larger half of residue classes modulo $(2m+2)$, say $m+2 \leq \lambda_{i_1}, \ldots, \lambda_{i_r} \leq 2m+2$, then we define $\tau_k = (i_k \;\; m+2-i_k)$ for $1 \leq k \leq r$ and also let $\tau = \tau_1 \cdots \tau_k$. Now we can easily calculate $A(\Lambda)$:

$$\begin{aligned}
A(\Lambda) &= \sum_{\sigma \in S_{m+1}} \operatorname{sgn}(\sigma) \prod_{k=1}^{r} \left(\frac{\zeta^{\sigma(i_k)-1}}{\zeta^{\sigma(m+2-i_k)-1}} \right)^{\lambda_{i_k}/2} \prod_{\substack{1 \leq i \leq s(m) \\ i \neq i_k \forall k}} \left(\frac{\zeta^{\sigma(i)-1}}{\zeta^{\sigma(m+2-i)-1}} \right)^{\lambda_i/2} \\
&= \sum_{\sigma \in S_{m+1}} \operatorname{sgn}(\sigma) \prod_{k=1}^{r} \left(\frac{\zeta^{\sigma\tau_k(i_k)-1}}{\zeta^{\sigma\tau_k(m+2-i_k)-1}} \right)^{-\lambda_{i_k}/2} \prod_{\substack{1 \leq i \leq s(m) \\ i \neq i_k \forall k}} \left(\frac{\zeta^{\sigma(i)-1}}{\zeta^{\sigma(m+2-i)-1}} \right)^{\lambda_i/2} \\
&= \sum_{\sigma \in S_{m+1}} (-1)^r \operatorname{sgn}(\sigma\tau) \prod_{k=1}^{r} \left(\frac{\zeta^{\sigma\tau(i_k)-1}}{\zeta^{\sigma\tau(m+2-i_k)-1}} \right)^{\lambda'_{i_k}/2} \prod_{\substack{1 \leq i \leq s(m) \\ i \neq i_k \forall k}} \left(\frac{\zeta^{\sigma\tau(i)-1}}{\zeta^{\sigma\tau(m+2-i)-1}} \right)^{\lambda'_i/2} \\
&= (-1)^r \sum_{\sigma\tau \in S_{m+1}} \operatorname{sgn}(\sigma\tau) \prod_{i=1}^{s(m)} \left(\frac{\zeta^{\sigma\tau(i)-1}}{\zeta^{\sigma\tau(m+2-i)-1}} \right)^{\lambda'_i/2} = (-1)^r A(\Lambda').
\end{aligned}$$

The third line follows because τ_k acts only on i_k and $(m+2-i_k)$ for each k, and τ is the product of these transpositions.

We have shown that if $\Lambda \notin S(m)$, then $A(\Lambda) = 0$, so these vectors are removed from the index of summation in (2.7). We have also shown that if $\Lambda \in S(m)$, then $A(\Lambda) = \pm A(\Lambda')$, with the sign determined by the parity of the number of vector components satisfying $\lambda_i > m+1$, and thus we define the subsets $S_{\pm}(m)$. Dividing (2.7), (2.5) and (2.6) by $c = \prod_{k=1}^{m}(1-\zeta^k)^{m+1-k}$ gives Theorems 1.1 and 1.2.

Q.E.D.

Remark. It is also possible to make the substitution $x_i = \zeta^i, 1 \leq i \leq m+1$ in (2.1) and obtain identities similar to those in Theorems 1.1 and 1.2. The results are essentially the same, with only minor adjustments to Definition 1.1.

3. Modular forms and the proof of Corollary 1.7

For a positive integer N, define the subgroup $\Gamma_0(N)$ of $SL_2(\mathbb{Z})$ as follows:

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\}. \tag{3.1}$$

Now suppose that $f(z)$ is a holomorphic function on the upper half of the complex plane and at the cusps of $\Gamma_0(N)$, and let χ be a Dirichlet character modulo N. We say that $f(z)$ is a modular form with character of weight k with respect to $\Gamma_0(N)$ if

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k f(z), \tag{3.2}$$

for all z in the upper half of the complex plane and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$. The finite dimensional space of such modular forms is denoted by $M_k(\Gamma_0(N), \chi)$.

If we let $q = e^{2\pi i z}$, we may construct a Fourier expansion for such a modular form:

$$f(z) = \sum_{n=0}^{\infty} a(n) q^n.$$

In fact, we may identify any form with its Fourier expansion.

Next, recall Dedekind's eta-function:

$$\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n). \tag{3.3}$$

A function $f(z)$ is called an eta-product if it can be expressed as a product of the form

$$f(z) = \prod_{\delta | N} \eta^{r_\delta}(\delta z), \tag{3.4}$$

where N and each r_δ are integers. Such functions have many unique properties. Newman ([N1],[N2]) proves that certain eta-products fulfill the functional equation (3.2) for modular forms with character:

Theorem 3.1. *(Newman) If $f(z) = \prod_{\delta|N} \eta^{r_\delta}(\delta z)$ is an eta-product for which*

$$\sum_{\delta|N} \delta r_\delta \equiv 0 \pmod{24} \tag{3.5}$$

and

$$\sum_{\delta|N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24}, \tag{3.6}$$

then $f(z)$ satisfies the functional equation (3.2) for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, where $k = \frac{1}{2}\sum_{\delta|N} r_\delta$. Here χ is the character defined by:

$$\chi(d) = \left(\frac{(-1)^k s}{d}\right) \text{ and } s = \prod_{\delta|N} \delta^{r_\delta}.$$

If an eta-product satisfies the functional equation (3.2), we still must demonstrate holomorphicity at the cusps of $\Gamma_0(N)$ to prove it is a modular form. The following observation is well known (for example, see Ligozat [L]):

Proposition 3.2. *Let c, d, and N be positive integers with $d|N$ and $(c,d) = 1$. With the notation as above, if the eta-product $f(z)$ satisfies (3.5) and (3.6), then the order of vanishing of $f(z)$ at the cusp $\frac{c}{d}$ is*

$$\frac{1}{24} \sum_{\delta|N} \frac{N(d,\delta)^2 r_\delta}{\left(d, \frac{N}{d}\right) d\delta}. \tag{3.7}$$

Using Theorem 3.1 and Proposition 3.2 we obtain the following:

Theorem 3.3. *If m is even, then*

$$\sum_{n=0}^{\infty} \left((P_+(n,m) - P_-(n,m)\right) q^{n/2} \in M_{m/2}(\Gamma_0(2(m+1)), \chi), \tag{3.8}$$

where $\chi(d) = \left(\frac{(-1)^{m/2}(m+1)}{d}\right)$.

If m is odd, then

$$\sum_{n=0}^{\infty} \left((P_+(n,m) - P_-(n,m)\right) q^{n} \in M_{(m+1)/2}(\Gamma_0(4(m+1)), \chi), \tag{3.9}$$

where $\chi(d) = \left(\frac{(-1)^{(m+1)/2}(m+1)}{d}\right)$.

Remark. Note that Corollary 1.6 implies that (3.8) is a power series in integer powers of q.

Proof of Theorem 3.3. We give the proof in the case that m is even; the proof for odd m is entirely similar. By Theorem 1.2, for even m,

$$\sum_{n=0}^{\infty} (P_+(n,m) - P_-(n,m))\, q^{n/2} = q^{(m^2+2m)/8} \prod_{n=1}^{\infty} \frac{(1-q^n)}{(1-q^{2n})^2} \frac{(1-q^{2(m+1)n})^{2m+2}}{(1-q^{(m+1)n})^{m+1}} \tag{3.10}$$

$$= \frac{q^{1/24}}{q^{4/24}} \frac{q^{4(m+1)^2/24}}{q^{(m+1)^2/24}} \prod_{n=1}^{\infty} \frac{(1-q^n)}{(1-q^{2n})^2} \frac{(1-q^{2(m+1)n})^{2m+2}}{(1-q^{(m+1)n})^{m+1}}$$

$$= \frac{\eta(z)}{\eta^2(2z)} \frac{\eta^{2m+2}(2(m+1)z)}{\eta^{m+1}((m+1)z)} := f(z).$$

If we let $N = 2(m+1)$, then we may write $f(z) = \prod_{\delta|N} \eta^{r_\delta}(\delta z)$ as in Theorem 3.1 and Proposition 3.2. We have

$$\sum_{\delta|N} \delta r_\delta = 1(1) - 2(2) + 4(m+1)^2 - (m+1)^2 = -3 + 3(m+1)^2$$

and

$$\sum_{\delta|N} \frac{N}{\delta} r_\delta = 2(m+1)\left(\frac{1}{1} - \frac{2}{2} + \frac{2(m+1)}{2(m+1)} - \frac{(m+1)}{(m+1)}\right) = 0 \equiv 0 \pmod{24}.$$

So $f(z)$ satisfies (3.5) and (3.6). Therefore, by Theorem 3.1, $f(z)$ satisfies (3.2) with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2(m+1))$, $k = \frac{m}{2}$, and $\chi(d) = \left(\frac{(-1)^{m/2}(m+1)}{d}\right)$.

Now we must check the holomorphicity of $f(z)$ at the cusps of $\Gamma_0(2(m+1))$. By Proposition 3.2, it suffices to show that the following is non-negative:

$$\frac{(d,1)^2(1)}{1} - \frac{(d,2)^2(2)}{2} + \frac{(d,2(m+1))^2(2(m+1))}{2(m+1)} - \frac{(d,(m+1))^2((m+1))}{(m+1)}.$$

If $2 \nmid d$, then

$$(d,1)^2 - (d,2)^2 + (d,2(m+1))^2 - (d,(m+1))^2 = 0.$$

If $2|d$, then

$$(d,1)^2 - (d,2)^2 + (d,2(m+1))^2 - (d,(m+1))^2 = 1 - 4 + 4(d,(m+1))^2 - (d,(m+1))^2 \geq 0.$$

Q.E.D.

To prove Corollary 1.7, we recall a theorem of Serre [S].

Theorem 3.4. *(Serre) Let $f(z)$ be a holomorphic modular form of positive integer weight k with character χ on $\Gamma_0(N)$ with Fourier expansion*

$$f(z) = \sum_{n=0}^{\infty} a(n) q^n \tag{3.11}$$

where $a(n)$ are algebraic integers in some number field and N is a positive integer. If M is a positive integer, then there exists a positive constant α such that there are $O\left(\frac{x}{log^{\alpha} x}\right)$ integers $n \leq x$ where the $a(n)$ are not divisible by M.

Proof of Corollary 1.7. By Theorem 3.3, for every positive integer k the generating function

$$\sum_{n=0}^{\infty} \left(P_+(n,k) - P_-(n,k)\right) q^n$$

is a holomorphic integer weight modular form. Therefore, Theorem 3.4 immediately implies Corollary 1.7.

Q.E.D.

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