ON q-DIFFERENCE EQUATIONS FOR PARTITIONS WITHOUT k-SEQUENCES

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Abstract. In his study of partitions without k-sequences, Andrews proved a double hypergeometric q-series representation of the generating series and a corresponding k-term q-difference equation. In this note we give new proofs of the double series formula, as well as a new two-term q-difference equation. In both cases, we provide independent analytic and combinatorial proofs.

1. Introduction and statement of results

A *partition* of a positive integer n is a non-increasing sequence of positive integers whose sum is n, and the number of partitions of n is traditionally denoted by $p(n)$ (a general reference is [3]). A *sequence* in a partition is any subsequence of consecutive integers that each appear as parts. If such a sequence consists of k consecutive integers, it is known as a *k-sequence.* In particular, any non-empty partition trivially contains a 1-sequence. The number of partitions that do *not* contain a k-sequence is denoted by $p_k(n)$, and we write the corresponding generating function as

$$G_k(q) := \sum_{n\geq 0} p_k(n)q^n.$$

Integer partitions without sequences were first studied by MacMahon [11]. In particular, he used partition conjugation to show that partitions without sequences are equinumerous with partitions in which all parts are repeated, with the possible exception of the largest part. This directly gives a hypergeometric q-series expression for the generating function, namely

$$G_2(q) = 1 + \sum_{n\geq 1} \frac{q^n \left(q^6;q^6\right)_{n-1}}{\left(1-q^n\right)\left(q^2;q^2\right)_{n-1}\left(q^3;q^3\right)_{n-1}}, \tag{1.1}$$

where the q-Pochhammer symbol is defined by $(a)_n = (a;q)_n := \prod_{j=0}^{n-1}(1-aq^j)$.

The family of partitions without k-sequences was introduced in [9], where they were of interest because of their relationship to probabilistic bootstrap percolation models. In particular, the *metastability threshold* of the k-cross neighborhood is intimately related to asymptotic estimates of $p_k(n)$; see [6] and [8] for further discussion of these approximations. In [5] the first and the third author considered the case of partitions without sequences and found an asymptotic series for $p_2(n)$ with polynomial error by amplifying the Hardy-Ramanujan Circle Method using a Saddle Point analysis. Indeed, this case is of particular interest in

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number theory due to an alternative form of (1.1) found by Andrews [2] that involves one of Ramanujan's mock theta functions as described in [12]. The most recent progress in the analytic study of partitions without k-sequences is due to Kane and Rhoades [10], who proved a refined form of an asymptotic formula conjectured by Andrews [2], by giving the general asymptotic formula

$$p_k(n) \sim \frac{1}{2k}\left(\frac{1}{6}\left(1-\frac{2}{k(k+1)}\right)\right)^{\frac{1}{4}}\frac{1}{n^{\frac{3}{4}}}\exp\left(\pi\sqrt{\frac{2}{3}\left(1-\frac{2}{k(k+1)}\right)n}\right). \tag{1.2}$$

They achieved this by analyzing a Markov-type process for generating partitions without k-sequences; see [7] for a related setting in which bootstrap percolation models and Markov-type processes arise. This asymptotic analysis is quite different from that used in [4], in which the Constant Term Method was used to determine the asymptotic enumeration of overpartitions with k-sequences. Strikingly, it can be checked that the analogous calculation fails in the case of partitions without k-sequences, so (1.2) cannot be proven in this way.

In this note we are primarily interested in the combinatorial theory of partitions without k-sequences. This aspect is also found in Andrews' study, as he evaluated all of the generating functions $G_k(q)$. In particular, Theorem 2 of [2] gives a uniform double q-hypergeometric series expansion for all $k \geq 1$:

$$G_k(q) = \frac{1}{(q;q)_\infty}\sum_{r,s\geq 0}\frac{(-1)^r q^{\frac{(k+1)k(r+s)^2}{2}+\frac{(k+1)(s+1)s}{2}}}{(q^k;q^k)_r\,(q^{k+1};q^{k+1})_s}. \tag{1.3}$$

Note that this formula includes the case $k=1$, where it is clear combinatorially that $G_1(q) = 1$.

Andrews' proof follows from the the theory of q-difference equations. To describe this, define

$$G_k(x) = G_k(x;q) := \sum_{n,m\geq 0} p_k(m,n)x^m q^n,$$

where $p_k(m,n)$ denotes the number of partitions of n into m parts that contain no k-sequences. Andrews showed that

$$G_k(x) = G_k(xq) + \sum_{j=1}^{k-1}\frac{x^j q^{\frac{j(j+1)}{2}} G_k\left(xq^{j+1}\right)}{(xq;q)_j}, \tag{1.4}$$

which can be derived as a combinatorial identity by conditioning on the length of the sequence that begins with the part 1 (the first term corresponds to the case that there is no such sequence). It is then a standard calculation to verify that

$$g_k(x;q) := \frac{1}{(xq;q)_\infty}\sum_{r,s\geq 0}\frac{x^{kr+(k+1)s}(-1)^r q^{\frac{(k+1)k(r+s)^2}{2}+\frac{(k+1)(s+1)s}{2}}}{(q^k;q^k)_r\,(q^{k+1};q^{k+1})_s} \tag{1.5}$$

satisfies the difference equation (1.4) and the same initial conditions, and is therefore the unique solution (cf. Lemma 1 of [1]). Setting $x=1$ then gives (1.3).

In fact, the q-difference equation (1.4) is useful not only for evaluating the generating function $G_k(q)$ in closed form, but also fundamentally underlies the techniques used in the asymptotic analysis of $p_k(n)$. Indeed, the Markov-type process found in [10] can be understood as the direct conversion of the k-term recurrence into a linear recurrence on a k-dimensional vector space (again, see [7] for a related example).

In light of the importance of the q-difference equation, this note was inspired by the observation that the double sum in (1.3) and k-term recurrence in (1.4) are counter to the general philosophy that an ℓ-fold summation should correspond to an ℓ-fold q-difference equation. Our first main result remedies this, as we give a new two-term q-difference equation for $G_k(x;q)$.

Theorem 1.1. *If $|q|<1, |x|\leq 1$, and $k\geq 1$, then*

$$G_k(x)=\frac{1}{1-xq}\,G_k(xq)-\frac{x^k q^{\frac{k(k+1)}{2}}}{(xq)_k}\,G_k\left(xq^{k+1}\right).$$

Remark. Although the coefficients of $G_k(x)$ are non-negative, this is not manifestly evident from either (1.3) or Theorem 1.1 (in contrast to (1.4)).

We provide two proofs of Theorem 1.1 in Section 2, beginning with an analytic derivation of our new q-difference equation from Andrews' original recurrence. This is followed by an independent combinatorial argument using an inclusion-exclusion enumeration of partitions without k-sequences. The note concludes in Section 3, where we provide two new proofs of the double hypergeometric q-series in (1.3). The first proof is again a direct algebraic calculation from the q-difference equation. For the second proof we employ an alternative combinatorial setting. Instead of partitions with without k-sequences, we consider partitions that can be decomposed into a disjoint collection of k and $(k+1)$-sequences, with the further condition that for each k-sequence, the next largest part does not occur in the partition.

2. Proof of q-difference equation

In this section we prove Theorem 1.1. We first use Andrews' original proof of (1.3) to derive the new two-term q-difference equation, but we then give a new combinatorial argument that is independent from Andrews' work.

2.1. **Analytic argument.** The q-difference equation for $G_k(x;q)$ is greatly simplified if one instead defines

$$F_k(x)=F_k(x;q):=(xq;q)_\infty G_k(x;q). \tag{2.1}$$

Theorem 1.1 follows immediately from a two-term q-difference equation for $F_k(x)$.

Proposition 2.1. *If $|q|<1, |x|\leq 1$, and $k\geq 2$, then*

$$F_k(x)-F_k(xq)=-x^k q^{\frac{k(k+1)}{2}}\left(1-xq^{k+1}\right)F_k\left(xq^{k+1}\right).$$

Proof. Multiplying through by $(xq)_\infty$, Andrews' q-difference equation from (1.4) yields

$$\begin{aligned}F_k(x)&=(xq)_\infty G_k(xq)+\sum_{j=1}^{k-1}x^j q^{\frac{j(j+1)}{2}}\left(xq^{j+1}\right)_\infty G_k\left(xq^{j+1}\right)\\&=(1-xq)F_k(xq)+\sum_{j=1}^{k-1}x^j q^{\frac{j(j+1)}{2}}\left(1-xq^{j+1}\right)F_k\left(xq^{j+1}\right).\end{aligned}$$

Rearranging terms then gives a convenient q-difference equation for $F_k(x)$, namely

$$F_k(x) - F_k(xq) = -xqF_k(xq) + \sum_{j=1}^{k-1} x^j q^{\frac{j(j+1)}{2}} \left(1 - xq^{j+1}\right) F_k\left(xq^{j+1}\right). \tag{2.2}$$

The goal is therefore to show that the right side of this expression collapses to a single term.

For this, we write the expansion in powers of x as

$$F_k(x) = \sum_{m\geq 0} f_m(q) x^m.$$

Abbreviating $f_m = f_m(q)$, equation (2.2) then becomes

$$\sum_{m\geq 0} (1-q^m) f_m x^m = \sum_{m\geq 0} \left(-x^{m+1} q^{m+1} f_m + \sum_{j=1}^{k-1} f_m \left(1 - xq^{j+1}\right) x^{m+j} q^{\frac{j(j+1)}{2}+m(j+1)}\right).$$

Isolating the coefficient of x^m, we thus have the equality

$$\begin{aligned}
(1-q^m) f_m &= -q^m f_{m-1} + \sum_{j=1}^{k-1} \left(q^{\frac{j(j+1)}{2}+(m-j)(j+1)} f_{m-j} - q^{(m-j-1)(j+1)+\frac{1}{2}(j+2)(j+1)} f_{m-j-1}\right) \\
&= -q^m \left(1-q^{m-1}\right) f_{m-1} - \sum_{j=2}^{k-1} q^{mj-\frac{j(j-1)}{2}} \left(1-q^{m-j}\right) f_{m-j} - q^{mk-\frac{k(k-1)}{2}} f_{m-k}. \qquad (2.3)
\end{aligned}$$

Note that the first term on the right hand side contains $(1-q^{m-1})f_{m-1}$, which can be recursively replaced again using (2.3). After doing so the inner sums cancel, and a bit of algebra yields the simple identity

$$(1-q^m) f_m = q^{m(k+1)-\frac{k(k+1)}{2}} \left(f_{m-k-1} - f_{m-k}\right). \tag{2.4}$$

Multiplying (2.4) by x^m and summing over m we recover the corresponding series identity

$$\begin{aligned}
F_k(x) - F_k(xq) &= \sum_{m\geq 0} \left(-q^{(m+k)(k+1)-\frac{k(k+1)}{2}} f_m x^{m+k} + q^{(m+k+1)(k+1)-\frac{k(k+1)}{2}} f_m x^{m+k+1}\right) \\
&= -x^k q^{\frac{k(k+1)}{2}} F_k\left(xq^{k+1}\right)\left(1 - xq^{k+1}\right),
\end{aligned}$$

which completes the proof. □

Proof of Theorem 1.1. Inserting Proposition 2.1 into the definition (2.1) gives

$$(xq)_\infty G_k(x) - (xq^2)_\infty G_k(xq) = -x^k q^{\frac{k(k+1)}{2}} \left(1 - xq^{k+1}\right)\left(xq^{k+2}\right)_\infty G_k\left(xq^{k+1}\right).$$

This is equivalent to the q-difference equation in the theorem statement. □

2.2. **Combinatorics of the difference equation.** Although we used Andrews' q-difference equation (1.4) at the beginning of our proof in Section 2.1, our new q-difference equation and the double hypergeometric series for $G_k(q)$ can also be proven independently. In this section we provide a second proof of Theorem 1.1 using combinatorial partition theory; the arguments in the previous section can then be derived in reverse order without appealing to Andrews' work.

Specifically, we prove Theorem 1.1 directly using an "inclusion-exclusion" type enumeration argument. Referring to the q-difference equation from the theorem statement, the left-hand side is by definition the two-variable generating function for all partitions without k-sequences. If all of the 1s in such a partition are removed, the remaining parts still form a partition without k-sequences. The first term on the right-hand side generates all such partitions, with an arbitrary number of 1s added to partitions without k-sequences that begin with a part of size 2 or larger. But this first term is an overcount, as it also includes partitions that contain a 1 in addition to the $(k-1)-$sequence $2, 3, \dots k$. Taken together, these parts form a k-sequence. The overcount is then corrected by subtracting the second term on the right, which generates all partitions with the k-sequence $1, 2, \dots, k$, no $k+1$, and no other k-sequences.

3. Proof of double summation formula

In this section we provide two new proofs of the double summation formula (1.3). In the first proof we verify algebraically that the two-variable double series satisfies our new q-difference equation from Theorem 1.1, and for the second proof we give a new combinatorial description in terms of a certain weighted enumeration of partitions into distinct parts.

3.1. **Analytic verification.** Our first proof is similar to that used by Andrews in [2] when verifying that the double series satisfies (1.4). In particular, our aim is now to show that

$$f_k(x) = f_k(x;q) := \sum_{r,s\geq 0} \frac{x^{kr+(k+1)s}(-1)^r\, q^{\frac{(k+1)k(r+s)^2}{2}+\frac{(k+1)(s+1)s}{2}}}{\left(q^k;q^k\right)_r \left(q^{k+1};q^{k+1}\right)_s}$$

satisfies the q-difference equation from Proposition 2.1. Specifically, we see that

$$f_k(x) - f_k(xq) = \sum_{r,s\geq 0} \frac{x^{kr+(k+1)s}(-1)^r\, q^{\frac{(k+1)k(r+s)^2}{2}+\frac{(k+1)(s+1)s}{2}}}{\left(q^k;q^k\right)_r \left(q^{k+1};q^{k+1}\right)_s}\left(1-q^{kr+(k+1)s}\right). \tag{3.1}$$

Rewriting

$$1-q^{kr+(k+1)s} = 1-q^{(k+1)s} + q^{(k+1)s}\left(1-q^{kr}\right)$$

and making the appropriate shifts in the summation indices, the right hand side of (3.1) simplifies to the desired expression

$$-x^k q^{\frac{k(k+1)}{2}}\left(1-xq^{k+1}\right) f_k\left(xq^{k+1}\right)$$

(see the proof of Theorem 2.3 in [4] for a similar calculation).

3.2. **Combinatorial derivation of double sum.** We close by giving a new proof of the double series representation of $F_k(x)$ in which we use partition combinatorics to analyze Proposition 2.1. We first rewrite the q-difference equation as

$$F_k(x) = F_k(xq) - x^k q^{\frac{k(k+1)}{2}}\left(1-xq^{k+1}\right) F_k\left(xq^{k+1}\right). \tag{3.2}$$

This can now be related to the generating function of certain partitions into distinct parts. In particular, let $\mathcal{P}_k$ denote the set of partitions that can be decomposed into a disjoint collection of k and $(k+1)$-sequences, with the further condition that for each k-sequence, the next largest part does not occur in the partition. It is also convenient to refine these sets by writing $\mathcal{P}_k(n)$ for the partitions of n in $\mathcal{P}_k$.

Such a partition λ is bijectively associated to a subsequence of natural numbers $(a_1, a_2, \ldots, a_j)$ with

$$a_{i+1} - a_i \geq k + 1$$

and each a_i being marked as *Type* k or $k+1$ depending on whether it is a k or $k+1$-sequence that begins at a_i. The weight of the partition is then defined as $w(\lambda) := \#\{i \mid a_i \text{ is Type } k\}$, and we also denote the number of parts by $\ell(\lambda)$ and its size by $|\lambda|$. Note that $\ell(\lambda) = j(k+1) - w(\lambda)$.

For example, if $k = 1$, the following partition is in $\mathcal{P}_1(77)$:

$$\lambda = 2 + 3 + 4 + 6 + 7 + 8 + 9 + 11 + 13 + 14. \tag{3.3}$$

The associated subsequence is $(a_1, a_2, a_3, a_4, a_5, a_6) = (2, 4, 6, 8, 11, 13)$, with corresponding Types $(2, 1, 2, 2, 1, 2)$; its weight is $w(\lambda) = 2$ and its length is $\ell(\lambda) = 10$.

With these definitions, (3.2) can now be understood as a combinatorial statement about such partitions.

Lemma 3.1. *Suppose that $k \geq 1$ and $F_k(x; q)$ is a double series that satisfies* (3.2) *and $F_k(0; 0) = 1$. Then it can be expanded as*

$$F_k(x) = F_k(x; q) = \sum_{m,n \geq 0} Q_k(m, n) x^m q^n,$$

where

$$Q_k(m, n) := \sum_{\substack{\lambda \in \mathcal{P}_k(n) \\ \ell(\lambda) = m}} (-1)^{w(\lambda)}.$$

Proof. Following (3.2), we see that a non-zero term in $F_k(x)$ is constructed by choosing either $-x^k q^{\frac{k(k+1)}{2}} F_k(xq^{k+1})$ or $x^{k+1} q^{\frac{(k+1)(k+2)}{2}} F_k(xq^{k+1})$. The first case corresponds to a Type k sequence beginning at 1; note that the exponents on x and q count the parts $1, 2, \ldots k$, that there is a weight of -1, and that if (3.2) is iteratively applied to $F_k(xq^{k+1})$, the next part is at least $k + 2$. The second case analogously corresponds to a Type $k + 1$ sequence beginning at 1, which completes the proof. $\square$

A further combinatorial analysis of Lemma 3.1 gives another, independent proof of the hypergeometric series representation of $F_k(x)$. Our combinatorial argument is illustrated by a simple example following the proof.

Proposition 3.2. *If $|q| < 1, |x| \leq 1$, and $k \geq 2$, then*

$$F_k(x) = \sum_{r,s \geq 0} \frac{x^{kr+(k+1)s} (-1)^r \, q^{\frac{(k+1)k(r+s)^2}{2} + \frac{(k+1)(s+1)s}{2}}}{(q^k; q^k)_r \, (q^{k+1}; q^{k+1})_s}.$$

Proof. Suppose that $\lambda \in \mathcal{P}_k(n)$ and $\ell(\lambda) = m$. Since this is a partition into distinct parts, if we remove the triangular partition $1 + 2 + \cdots + m$, then we are left with a new partition λ'. We thus obtain a new subsequence $(a'_1, \ldots, a'_m)$ such that $0 \leq a'_1$ and $a'_i \leq a'_{i+1}$, with strict inequality if a_i is Type k. Furthermore, each k or $(k + 1)$-sequence in λ becomes, respectively, a grouping of k or $k + 1$ identical parts in λ'.

Taken together, this means that λ' decomposes into a partition into distinct multiples of k and an ordinary partition into multiples of $k + 1$ (where parts of size 0 are allowed). The

q-binomial theorem (see Corollary 2.2 in [1]) then gives a double series for the generating function

$$\sum_{\lambda\in\mathcal{P}_k}(-1)^{w(\lambda)}x^{\ell(\lambda)}q^{|\lambda'|}=\frac{\left(x^k;q^k\right)_\infty}{\left(x^{k+1};q^{k+1}\right)_\infty}=\sum_{r,s\geq 0}\frac{(-1)^r x^{kr+(k+1)s}q^{\frac{kr(r-1)}{2}}}{\left(q^k;q^k\right)_r\left(q^{k+1};q^{k+1}\right)_s}. \tag{3.4}$$

Note that in the double series, each λ is associated to a pair (r,s), where r is the number of a_i of Type k, s is the number of a_i of Type $k+1$, and $\ell(\lambda)=kr+(k+1)s$.

To complete the proof, observe that

$$|\lambda|=|\lambda'|+\frac{\ell(\lambda)(\ell(\lambda)+1)}{2}.$$

Recalling Lemma 3.1 and plugging in to (3.4), we obtain

$$F_k(x)=\sum_{r,s\geq 0}\frac{(-1)^r x^{kr+(k+1)s}q^{\frac{kr(r-1)}{2}+\frac{(kr+(k+1)s)(kr+(k+1)s+1)}{2}}}{(q^k;q^k)_r(q^{k+1};q^{k+1})_s}.$$

Simplifying the exponent gives the proposition statement. □

Remark. For the example from (3.3), the associated λ' is $1+1+1+2+2+2+2+3+4+4$, which decomposes into the partition into distinct parts $1+3$ and the partition into even parts $2+4+4+8$.

Remark. Similar to the discussion of (1.5), the solution to (3.2) is unique subject to the assumption that the series $F_k(x)$ begins with 1.

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