HIGH DENSITY PIECEWISE SYNDETICITY OF SUMSETS

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Abstract. Renling Jin proved that if A and B are two subsets of the natural numbers with positive Banach density, then $A+B$ is piecewise syndetic. In this paper, we prove that, under various assumptions on positive lower or upper densities of A and B, there is a high density set of witnesses to the piecewise syndeticity of $A+B$. The key technical tool is a Lebesgue density theorem for measure spaces induced by cuts in the nonstandard integers.

1. Introduction and preliminaries

1.1. **Sumsets and piecewise syndeticity.** The earliest result on the relationships between density of sequences, sum or difference sets, and syndeticity is probably Furstenberg's theorem mentioned in [7, Proposition 3.19]: If A has positive upper Banach density, then $A-A$ is syndetic, i.e. has bounded gaps. The proof of the theorem is essentially a pigeonhole argument.

In [9] Jin shows that if A and B are two subsets of $\mathbb{N}$ with positive upper Banach densities, then $A+B$ must be piecewise syndetic, i.e. for some m, $A+B+[0,m]$ contains arbitrarily long intervals. Jin's proof uses nonstandard analysis. In [11], this result is extended to abelian groups with tiling structures. In [9, 11] the question as to whether this result can be extended to any countable amenable group is posed, and in [2] a positive answer to the above question is proven. It is shown that if A and B are two subsets of a countable amenable group with positive upper Banach densities, then $A\cdot B$ is piecewise Bohr, which implies piecewise syndeticity. In fact, a stronger theorem is obtained in the setting of countable abelian groups: A set S is piecewise Bohr if and only if S contains the sum of two sets A and B with positive upper Banach densities. Jin's theorem was generalized to arbitrary amenable groups in [5]. At the same time, several new proofs of the theorem in [9] have appeared. For example, an ultrafilter proof is obtained in [1]. A more quantitative proof that includes a bound based

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on the densities is obtained in [4] by nonstandard methods, and in [3] by elementary means.

However, there has not been any progress on extending the theorem in [9] to lower asymptotic density or upper asymptotic density instead of upper Banach density. Of course, if A and B have positive lower (upper) asymptotic densities then they have positive Banach density, so $A+B$ must be piecewise syndetic. In this paper we show that there is significant uniformity to the piecewise syndetiticy in the sense that there are a large density of points in the sumset with no gap longer than some fixed m. Furthermore this result can be extended to all finite dimensions. Specifically we show that if $A \subseteq \mathbb{Z}^d$ has positive upper d-dimensional asymptotic density α, and B simply has positive Banach density, then there exists a fixed m such that for all k the upper density of the set of elements z in $\mathbb{Z}^d$ such that $z+[-k,k]^d \subseteq A+B+[-m,m]^d$ is at least α. For lower density we show that the analogous conclusion must be slightly weakened as follows: If $A \subseteq \mathbb{Z}^d$ has positive lower d-dimensional asymptotic density α, and B has positive Banach density, then for any $\epsilon > 0$ there exists a fixed m such that for all k the lower density of the set of elements z in $\mathbb{Z}^d$ such that $z+[-k,k]^d \subseteq A+B+[-m,m]^d$ is at least $\alpha-\epsilon$. If both A and B have positive lower density in one dimension then, by using Mann's Theorem, we show that the set of elements z in $\mathbb{Z}$ such that $z+[-k,k] \subseteq A+B+[-m,m]$ has a strong version of upper density of at least $\alpha+\beta$.

The nonstandard methods used in this paper include a new Lebesgue Density Theorem for "cuts" in the nonstandard integers. In [13] a quasi-order-topology, with respect to each additive cut, was defined on a hyperfinite interval $[0,H]$ of integers. Motivated by the duality* of the ideal of null sets and the ideal of meager sets of real numbers, and the fact that the sum of two sets with positive Lebesgue measure can never be meager (because it always contains a non-empty open interval), a question was raised in [13]: Is the sum of any two sets with positive Loeb measure in a hyperfinite interval $[0,H]$ non-meager in the sense of the quasi-order-topology? A positive answer to the question above led to Jin's result about piecewise syndeticity. Here we study these cuts in d dimensions and prove the following result: If $H \in {}^*\mathbb{N} \setminus \mathbb{N}$, U is a subset of $[1,H]$ that is closed under addition, $\mathcal{U} = (-U) \cup \{0\} \cup (U)$, and E is an internal subset of $[-H,H]^d$ then almost all points x in $E+\mathcal{U}^d$ are points of density in the sense that

$$\liminf_{\nu > U} \mu_{x+[-\nu,\nu]^d}\left(\left(E+\mathcal{U}^d\right) \cap (x+[-\nu,\nu]^d)\right) = 1,$$

*The ideal of null sets $\mathcal{N}$ is the collection of all subsets of $\mathbb{R}$ with Lebesgue measure 0 and the ideal of meager sets $\mathcal{M}$ is the collection of all meager subsets of $\mathbb{R}$, where a set is called meager if it is a countable union of nowhere dense sets. $\mathcal{N}$ and $\mathcal{M}$ are dual ideals in the sense that $\mathbb{R}$ is the union of a meager set and a null set.

or, equivalently, to clarify the meaning of lim inf in this setting:

$$\sup_{\xi > U} \inf_{U < \nu < \xi} \mu_{x+[-\nu,\nu]^d} \left(\left(E + \mathcal{U}^d \right) \cap (x + [-\nu, \nu]^d) \right) = 1,$$

where $\mu_{x+[-\nu,\nu]^d}$ is the Loeb measure on $x + [-\nu, \nu]^d$. Here, as in the rest of the paper, when we write that an element is greater than an initial segment we mean that it is larger than every element in that segment. For example $U < \nu$ means that for all $u \in U$, $u < \nu$. We use the density theorem above in the case that $U = \mathbb{N}$ to obtain many of the aforementioned standard results.

A word about notation: In an effort to clarify standard vs. nonstandard sets and elements, we will reserve H, I, J, K, L, M, N for infinite hypernatural numbers, while ν, ξ, ζ will denote (possibly standard) hypernatural numbers; Lower case letters denote elements of $\mathbb{Z}$ or $\mathbb{Z}^d$ and their nonstandard extensions; A and B will be reserved for standard subsets of $\mathbb{N}$ (we do not include 0 in $\mathbb{N}$), $\mathbb{Z}$, or $\mathbb{Z}^d$; E,R, S, T, X and Y will be used for subsets of ${}^*\mathbb{Z}^d$, with E only used for internal sets. If $(a_n)_{n \in \mathbb{N}}$ is a sequence, and ν is an infinite hypernatural number, we denote by a_ν the value at ν of the nonstandard extension of the sequence $(a_n)_{n \in \mathbb{N}}$. We use μ for measure, d for density functions, and d for dimension. Here the values of d are only natural numbers. Despite these conventions the location of elements and sets is usually noted at the time, at the risk of redundancy, but in the interest of clarity.

1.2. **Standard concepts of density and structure on sequences.** In this paper we consider the following notions of density for a subset A of $\mathbb{Z}^d$:

- the *lower (asymptotic) density*
$$\underline{\mathrm{d}}(A) := \liminf_{n \to \infty} \frac{|A \cap [-n, n]^d|}{(2n+1)^d};$$
- the *upper (asymptotic) density*
$$\overline{\mathrm{d}}(A) := \limsup_{n \to \infty} \frac{|A \cap [-n, n]^d|}{(2n+1)^d};$$
- the *Schnirelmann density*
$$\sigma(A) := \inf_{n} \frac{|A \cap [-n, n]^d|}{(2n+1)^d};$$
- the *(upper) Banach density*
$$\mathrm{BD}(A) := \lim_{n \to \infty} \sup_{x \in \mathbb{Z}^d} \frac{|A \cap (x + [-n, n]^d)|}{(2n+1)^d}.$$

In the particular case of $d = 1$ these are the usual notions of density for sequences of integers. It follows immediately from the definition that for any $\epsilon > 0$ there exists an $m \in \mathbb{N}$ such that

$$\sigma(A \cup [-m, m]^d) \geq \overline{\mathrm{d}}(A) - \epsilon.$$

Moreover it is useful to note that if $\mathrm{BD}(A) > 0$ then for any $\epsilon > 0$ there exists $m \in \mathbb{N}$ such that

$$\mathrm{BD}(A + [-m, m]^d) > 1 - \epsilon.$$

When $d = 1$ this is the content of Theorem 3.8 in [8].

We will refer to the following combinatorial notions of largeness for a subset A of $\mathbb{Z}^d$:

- A is *syndetic* iff there exists $m \in \mathbb{N}$ such that $A + [-m, m]^d = \mathbb{Z}^d$;
- A is *thick* iff there are arbitrarily large hypercubes completely contained in A, i.e. for all $k \in \mathbb{N}$ there exists $z \in \mathbb{Z}^d$ such that
$$z + [-k, k]^d \subseteq A;$$
- A is *piecewise syndetic* iff there exists $m \in \mathbb{N}$ such that $A+[-m, m]^d$ is thick, i.e. for all $k \in \mathbb{N}$ there exists $z \in \mathbb{Z}^d$ such that
$$z + [-k, k]^d \subseteq A + [-m, m]^d.$$
Thus, A is piecewise syndetic iff it is the intersection of a syndetic set and a thick set.

While defining the densities on sets of the form $[-n, n]^d$ is natural, all of our results involving the notion of upper or lower syndeticity can be easily adapted to the setting where one considers arbitrary Følner sequences. Of particular interest for all of our results is the case in which $d = 1$ where the interval $[-n, n]$ is replaced by $[1, n]$. This is the classical setting for the study of densities of subsets of natural numbers. To underscore the importance of that case and to improve clarity, almost all of our examples are specific to this case, although all theorems and proofs will be given in d dimensions wherever possible.

It is not difficult to show that $\mathrm{BD}(A) = 1$ iff A is thick; more precisely, if for some $k \in \mathbb{N}$ every cube $z + [-k, k]^d$ is not contained in A, then $\mathrm{BD}(A) \leq \frac{(2k+1)^d-1}{(2k+1)^d}$. On the other hand, for every $r < 1$ there exist sets of lower density at least r that are not piecewise syndetic. Indeed, if n is sufficiently large and

$$B = \bigcup_{j=n}^{\infty} \bigcup_{x \in \mathbb{Z}^d \setminus \{0\}} \left((j!)x + [1, (j-1)!]^d \right),$$

then $\mathbb{Z}^d \setminus B$ is an example of such a set.

1.3. **Nonstandard preliminaries.** We use *nonstandard analysis* to derive our results and we assume that the reader is familiar with elementary nonstandard arguments. For an introduction to nonstandard methods aimed specifically toward applications to combinatorial number theory see [10]. Throughout this paper, we always work in a countably saturated nonstandard universe.

We make extensive use of the concept of Loeb measure. Here we will always be starting with the counting measure on some internal subset E of

$[-H,H]^d$ where H is some element in ${}^*\mathbb{N}\setminus\mathbb{N}$. Often E itself is $[-H,H]^d$, but it may also be a set of the form $x+[-J,J]^d$ where $x\in\mathbb{Z}^d$ and $J\in {}^*\mathbb{N}\setminus\mathbb{N}$. For every internal D contained in E, the measure of D relative to E is defined to be $\mu_E(D):=\mathrm{st}(\frac{|D|}{|E|})$, where st is the standard part mapping. This defines a finitely additive measure on the algebra of internal subsets of E, which canonically extends to a countably additive probability measure on the σ-algebra of *Loeb measurable* subsets of E, and we will also write μ_E for this extension. If D is defined on a larger set than E then we will write simply $\mu_E(D)$ for $\mu_E(D\cap E)$.

We will make frequent use of the well-known proposition below, which gives nonstandard equivalents for the standard density properties. Proofs are included for convenience.

Proposition 1.1. *If A is a subset of $\mathbb{Z}^d$ then we have the following nonstandard equivalents of the standard asymptotic densities:*

(1) *if $\overline{\mathrm{d}}(A)\geq\alpha$ then for all $K\in {}^*\mathbb{N}\setminus\mathbb{N}$ there exists an $H\in {}^*\mathbb{N}\setminus\mathbb{N}$ such that $H<K$ and $\mu_{[-H,H]^d}({}^*A)\geq\alpha$. Conversely If there exists $H\in {}^*\mathbb{N}\setminus\mathbb{N}$ such that $\mu_{[-H,H]^d}({}^*A)\geq\alpha$ then $\overline{\mathrm{d}}(A)\geq\alpha$;*
(2) *$\underline{\mathrm{d}}(A)\geq\alpha$ iff for all $H\in {}^*\mathbb{N}\setminus\mathbb{N}$ $\mu_{[-H,H]^d}({}^*A)\geq\alpha$;*
(3) *If $\mathrm{BD}(A)\geq\alpha$ then for all $K\in {}^*\mathbb{N}\setminus\mathbb{N}$ there exists $J\in[0,K]\setminus\mathbb{N}$ and $x\in[-K,K]^d$ such that $\mu_{x+[-J,J]^d}({}^*A)\geq\alpha$. Conversely if there exists $J\in {}^*\mathbb{N}\setminus\mathbb{N}$ and $x\in {}^*\mathbb{Z}^d$ such that $\mu_{x+[-J,J]^d}({}^*A)\geq\alpha$, then $\mathrm{BD}(A)\geq\alpha$.*

Proof.

(1) If $\overline{\mathrm{d}}(A)\geq\alpha$ then there exists a sequence $n_i\to\infty$ such that for all $i\in\mathbb{N}$
$$\frac{|A\cap[-n_i,n_i]^d|}{(2n_i+1)^d}\geq\alpha-\frac{1}{i}.$$
Pick an infinite hypernatural number J such that $n_J<K$, and observe that $\mu_{[-n_J,n_J]}({}^*A)\geq\alpha$. Conversely suppose that there exists an $H\in {}^*\mathbb{N}\setminus\mathbb{N}$ such that $\mu_{[-H,H]^d}({}^*A)\geq\alpha$. Given any $\epsilon>0$ and any $m\in\mathbb{N}$ one can deduce by transfer that there is a natural number $n>m$ such that
$$\frac{|A\cap[-n,n]^d|}{(2n+1)^d}\geq\alpha-\epsilon.$$
Therefore $\overline{\mathrm{d}}(A)\geq\alpha$.
(2) $\underline{\mathrm{d}}(A)\geq\alpha$ iff for any $\epsilon>0$ there exists $n_\epsilon\in\mathbb{N}$ such that for all $n>n_\epsilon$
$$\frac{|A\cap[-n,n]^d|}{(2n+1)^d}\geq\alpha-\epsilon.$$

By transfer, this is true iff for all $H \in {}^*\mathbb{N}\setminus\mathbb{N}$ and every standard $\epsilon > 0$

$$\frac{|{}^*A \cap [-H,H]^d|}{(2H+1)^d} \geq \alpha - \epsilon,$$

which is equivalent to $\mu_{[-H,H]^d}({}^*A) \geq \alpha$.

(3) If $\mathrm{BD}(A) \geq \alpha$ then there exists a sequence j_i, with $j_i \to \infty$ and points $x_i \in \mathbb{Z}^d$ such that for all $i \in \mathbb{N}$

$$\frac{|A \cap (x_i + [-j_i, j_i]^d)|}{(2j_i+1)^d} \geq \alpha - 1/i.$$

We can pick an infinite hypernatural number L such that $j_L < K$ and $x_L < K$. We may now let $J = j_L$ and $x = x_L$. Conversely if there exist $J \in {}^*\mathbb{N}\setminus\mathbb{N}$ and $x \in {}^*\mathbb{Z}^d$ such that $\mu_{x+[-J,J]^d}({}^*A) \geq \alpha$, then given any $\epsilon > 0$ and any $m \in \mathbb{N}$ one can deduce by transfer that there is are natural numbers j and x such that $j > m$ and

$$\frac{|A \cap (x + [-j,j]^d)|}{(2j+1)^d} \geq \alpha - \epsilon.$$

This witnesses the fact that $\mathrm{BD}(A) \geq \alpha$.

□

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2. Points of Density

If E is an internal subset of ${}^*\mathbb{Z}^d$ and $x \in {}^*\mathbb{Z}^d$ define

$$\begin{aligned} d_E(x) &= \liminf_{\nu > \mathbb{N}} \mu_{x+[-\nu,\nu]^d}\left(\left(E + \mathbb{Z}^d\right) \cap (x + [-\nu,\nu]^d)\right) \\ &= \sup_{H > \mathbb{N}} \inf_{\mathbb{N}<\nu<H} \mu_{[-\nu,\nu]^d}\left(\left((E - x) + \mathbb{Z}^d\right) \cap [-\nu,\nu]^d\right). \end{aligned}$$

Definition 1. *If $d_E(x) = 1$ we say that x is a* point of density *of E, and we write $\mathcal{D}_E$ for the set of points of density of E.*

We note that $\mathcal{D}_E$ is not, in general, internal and that $\mathcal{D}_E + \mathbb{Z}^d \subseteq \mathcal{D}_E$. It is easy to see by countable saturation that for $r \in [0,1]$ and $x \in {}^*\mathbb{Z}^d$, $d_E(x) \geq r$ if and only if there is $H > \mathbb{N}$ such that for every $\mathbb{N} < \nu < H$

$$\mu_{x+[-\nu,\nu]^d}\left(\left(E + \mathbb{Z}^d\right) \cap (x + [-\nu,\nu]^d)\right) \geq r.$$

Theorem 2.1 is our main result concerning points of density, and can be regarded as an analogue of the Lebesgue density theorem. It implies, in particular, that an internal set of positive Loeb measure relative to some interval always has points of density.

Theorem 2.1. *If* $E \subseteq [-H,H]^d$ *is internal then* $\mathcal{D}_E$ *is Loeb measurable, and* $\mu_{[-H,H]^d}(\mathcal{D}_E) = \mu_{[-H,H]^d}(E+\mathbb{Z}^d)$.

In Section 6 we will consider a similar notion of density point for arbitrary cuts and prove, in Corollary 6.5, a more general version of Theorem 2.1, which can be regarded as a Lebesgue density theorem for measure spaces induced by cuts in the nonstandard integers.

It is worth noting that the Loeb measure in the usual sense does not satisfy a similar analogue of the Lebesgue density theorem. For example the set of even numbers smaller than H has relative Loeb measure $1/2$ on every infinite interval. Theorem 2.1 says that if we identify points that are a finite distance apart, then the Loeb measure on that quotient space does have a density theorem very similar to that of the Lebesgue measure. Proposition 2.2 highlights a way in which the theorem is even stronger than it is for Lebesgue measure, where sets might have no interval about a point of density that actually achieves relative measure 1.

Proposition 2.2. *If* E *is an internal subset of* $[-H,H]^d$ *then* $d_E(x)=1$ *if and only if there exists* $\nu > \mathbb{N}$ *such that for all* $\mathbb{N} < K < \nu$,

$$\mu_{x+[-K,K]^d}\left(\left(E+\mathbb{Z}^d\right)\cap(x+[-K,K]^d)\right)=1.$$

Proof. Suppose that $d_E(x)=1$. Then from the definition there exist $\nu_j > \mathbb{N}$ such that for all $\mathbb{N} < K < \nu_j$,

$$\mu_{x+[-K,K]^d}\left(\left(E+\mathbb{Z}^d\right)\cap(x+[-K,K]^d)\right)\geq 1-1/j.$$

By countable saturation we can find $\nu > \mathbb{N}$ less than all the ν_j. The converse is immediate. □

We can characterize points of density in terms of standard density functions on subsets of $\mathbb{Z}^d$ that are centered around nonstandard points. In order to do so we need to approximate the "$+\mathbb{Z}^d$" part of the statement by considering how a set intersects with larger and larger "blocks."

Given a subset A of $\mathbb{Z}^d$ (or an internal subset of ${}^*\mathbb{Z}^d$) and $n \in \mathbb{N}$ (or ${}^*\mathbb{N}$) we define the *n-block sets* $A_{[n]}$ and $A^{[n]}$ *of* A by

$$x \in A_{[n]} \text{ iff } (nx+[0,n-1]^d)\cap A \neq \varnothing.$$

and

$$A^{[n]} = nA_{[n]} + [0,n-1]^d.$$

We note that $A_{[n]}$ and $A^{[n]}$ have the same asymptotic densities, but are "scaled" differently, with $A^{[n]}$ on the same scale as A. In fact, $A \subseteq A^{[n]}$, which consists of a union of $[0,n-1]^d$ blocks whose position is determined by the elements of $A_{[n]}$. More specifically

$$x \in A_{[n]} \text{ iff } (nx+[0,n-1]^d)\cap A \neq \varnothing \text{ iff } nx+[0,n-1]^d \subseteq A^{[n]}.$$

Thus, blocks of the form $nx + [0, n-1]^d$ containing any element of A are "completely filled in" to form $A^{[n]}$.

If E is internal the set $E + \mathbb{Z}^d$ is, in general, external, but its properties can often be approximated by the internal sets $E^{[n]}$ or $E + [-n, n]^d$ for large finite n or "small" elements of $^*\mathbb{N} \setminus \mathbb{N}$. The following observations are all stratightforward and will be useful.

- If $j \in \mathbb{N}$ and $J \in {}^*\mathbb{N} \setminus \mathbb{N}$ and $E \subseteq [-H, H]^d$ is internal, then
$$\mu_{[-H,H]^d}(E^{[j]}) \leq \mu_{[-H,H]^d}(E + \mathbb{Z}^d) \leq \mu_{[-H,H]^d}(E^{[J]}).$$
- For any internal $E \subseteq [-H, H]^d$
$$\lim_{i\to\infty}(\mu_{[-H,H]^d}(E^{[i]})) = \lim_{i\to\infty}(\mu_{[-H,H]^d}(E + [-i, i]^d)) = \mu_{[-H,H]^d}(E + \mathbb{Z}^d).$$
- For $A \subseteq \mathbb{Z}^d$
$$\lim_{i\to\infty}(\overline{\mathrm{d}}(A_{[i]})) = \lim_{i\to\infty}(\overline{\mathrm{d}}(A^{[i]})) = \lim_{i\to\infty}(\overline{\mathrm{d}}(A + [-i, i]^d)),$$
and
$$\lim_{i\to\infty}(\underline{\mathrm{d}}(A_{[i]})) = \lim_{i\to\infty}(\underline{\mathrm{d}}(A^{[i]})) = \lim_{i\to\infty}(\underline{\mathrm{d}}(A + [-i, i]^d)).$$

The limits in the statement above always exist, even though it is not true that $i < j$ implies that the upper and lower densities of $A^{[j]}$ are at least those of $A^{[i]}$. For example in one dimension, if A consists of all numbers that are 0,1, or 2 mod 6 then $\underline{\mathrm{d}}(A_{[2]}) = \overline{\mathrm{d}}(A_{[2]}) = 2/3$ while $\underline{\mathrm{d}}(A_{[3]}) = \overline{\mathrm{d}}(A_{[3]}) = 1/2$. However, it is easy to see that for all i and for all $\epsilon > 0$ there exists l such that for all $j > l$, $\underline{\mathrm{d}}(A_{[j]}) \geq \underline{\mathrm{d}}(A_{[i]}) - \epsilon$ and $\overline{\mathrm{d}}(A_{[j]}) \geq \overline{\mathrm{d}}(A_{[i]}) - \epsilon$. Thus $\lim_{i\to\infty}(\underline{\mathrm{d}}(A_{[i]}))$ and $\lim_{i\to\infty}(\overline{\mathrm{d}}(A_{[i]}))$ always exist. Of course, $\lim_{i\to\infty}(\overline{\mathrm{d}}(A + [-i, i]^d))$ clearly always exists.

Proposition 2.3. *Let r be a standard real number between* 0 *and* 1*, and E be an internal subset of $^*\mathbb{Z}^d$. If $x \in {}^*\mathbb{Z}^d$, then $d_E(x) > r$ if and only if $\sigma((E-x)^{[n]} \cap \mathbb{Z}^d) > r$ for some $n \in \mathbb{N}$. In particular x is a point of density if and only if for every $r \in (0,1)$ there is $n \in \mathbb{N}$ such that $\sigma\left((E-x)^{[n]} \cap \mathbb{Z}^d\right) > r$.*

Proof. Suppose that $\sigma((E-x)^{[n]} \cap \mathbb{Z}^d) \leq r$ for every $n \in \mathbb{N}$. Fix an arbitrary strictly positive standard real number ϵ, and pick a sequence $(l_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$ such that
$$\frac{\left|(E-x)^{[n]} \cap [-l_n, l_n]^d\right|}{(2l_n + 1)^d} < r + \epsilon$$
for every $n \in \mathbb{N}$. Observe that $(l_n)_{n\in\mathbb{N}}$ is a divergent sequence of natural numbers. Let H be an arbitrary infinite hypernatural number, and pick an

infinite hypernatural number ν such that $l_\nu < H$. We have that

$$\mu_{[-l_\nu,l_\nu]^d}\left(E-x+\mathbb{Z}^d\right) \le \mu_{[-l_\nu,l_\nu]^d}\left((E-x)^{[\nu]}\right) \approx \frac{\left|(E-x)^{[\nu]}\cap[-l_\nu,l_\nu]^d\right|}{(2l_\nu+1)^d} < r+\epsilon.$$

Since ν could be an arbitrarily small element in ${}^*\mathbb{N}\setminus\mathbb{N}$, this shows that $d_E(x) \le r+\epsilon$. Being this true for every standard real number ϵ, $d_E(x) \le r$. Conversely suppose that $\sigma((E-x)^{[n]}) \ge r+\epsilon$ for some strictly positive standard real number ϵ. Thus for every $k \in \mathbb{N}$

$$\frac{\left|(E-x)^{[n]}\cap[-k,k]^d\right|}{(2k+1)^d} \ge r+\epsilon.$$

If H is an infinite hypernatural number, then by overspill there is an infinite $\nu < H$ such that

$$\mu_{[-\nu,\nu]^d}\left((E-x)+\mathbb{Z}^d\right) \ge \frac{\left|(E-x)^{[n]}\cap[-\nu,\nu]^d\right|}{(2\nu+1)^d} \ge r+\epsilon,$$

witnessing the fact that $d_E(x) \ge r+\epsilon$. □

Definition 2. *If E is an internal subset of ${}^*\mathbb{Z}^d$ and $x \in {}^*\mathbb{Z}^d$ we say that x is a* point of syndeticity *of E iff there exists a finite m such that $x+\mathbb{Z}^d \subseteq E+[-m,m]^d$.*

Equivalently, since $E+[-m,m]^d$ is internal, x is a point of syndeticity of E iff there exists $m\in\mathbb{N}$ and $K \in {}^*\mathbb{N}\setminus\mathbb{N}$ such that $x+[-K,K]^d \subseteq E+[-m,m]^d$. We will write $\mathcal{S}_E$ for the set of all points of syndeticity of E. Like $\mathcal{D}_E$, $\mathcal{S}_E$ is, in general, not internal, and $\mathcal{S}_E+\mathbb{Z}^d \subseteq \mathcal{S}_E$.

Theorem 2.4. *Suppose that X, Y are internal subsets of ${}^*\mathbb{Z}^d$ and $a, b \in {}^*\mathbb{Z}^d$. If $d_X(a) = 1$ and $d_Y(b) = 1$ then $a+b$ is a point of syndeticity of $X+Y$.*

Proof. By countable saturation there exists $\nu > \mathbb{N}$ such that

$$\mu_{[-\nu,\nu]^d}\left(\left(X-a+\mathbb{Z}^d\right)\cap[-\nu,\nu]^d\right) = 1$$

and

$$\mu_{[-\nu,\nu]^d}\left(\left(-Y+b+\mathbb{Z}^d\right)\cap[-\nu,\nu]^d\right) = 1.$$

If $\xi \in \left[-\frac{\nu}{2},\frac{\nu}{2}\right]^d$ then

$$\mu_{[-\nu,\nu]^d}\left(\left(-Y+b+\xi+\mathbb{Z}^d\right)\cap[-\nu,\nu]^d\right) \ge \frac{1}{2^d}$$

and hence

$$\left(-Y+b+\xi+\mathbb{Z}^d\right)\cap\left(X-a+\mathbb{Z}^d\right) \ne \varnothing.$$

This means that there are $x\in X$, $y \in Y$ and $u, v \in \mathbb{Z}^d$ such that

$$a+b+\xi+u = x+y+v.$$

This shows that

$$a+b+\left[-\frac{\nu}{2},\frac{\nu}{2}\right]^d \subset X+Y+\mathbb{Z}^d.$$

Thus

$$a+b+\left[-\frac{\nu}{2},\frac{\nu}{2}\right]^d$$

is contained in the increasing union of internal sets

$$\bigcup_{m\in\mathbb{N}}(X+Y+[-m,m]^d).$$

By saturation, it follows that

$$a+b+\left[-\frac{\nu}{2},\frac{\nu}{2}\right]^d \subseteq X+Y+[-m,m]^d$$

for some $m\in\mathbb{N}$. Now

$$a+b+\mathbb{Z}^d \subseteq a+b+\left[-\frac{\nu}{2},\frac{\nu}{2}\right]^d \subseteq X+Y+[-m,m]^d,$$

and hence $a+b$ is a point of syndeticity of $X+Y$. □

Proposition 2.5. *Let E be an internal subset of $[-H,H]^d$. Then $\mathcal{S}_E$ is $\mu_{[-H,H]^d}$-measurable. Moreover, if $\mu(\mathcal{S}_E)=\alpha>0$ then for all (standard) $\epsilon>0$ there exists a (standard) $m\in\mathbb{N}$ such that for all (standard) $k\in\mathbb{N}$,*

$$\mu_{[-H,H]^d}(\{z\in[-H,H]^d : z+[-k,k]^d\subseteq E+[-m,m]^d\})\geq \alpha-\epsilon.$$

Proof. For each $i\in\mathbb{N}$ let

$$\mathcal{S}_E^i=\left\{x\in\ [-H,H]^d:\ x+\mathbb{Z}^d\subseteq E+[-i,i]^d\right\}.$$

Then

$$\mathcal{S}_E=\bigcup_{i=1}^{\infty}\mathcal{S}_E^i.$$

Each $\mathcal{S}_E^i$ is measurable since

$$\mathcal{S}_E^i=\bigcap_{z\in\mathbb{Z}^d}(E+[-i,i]^d+z),$$

and so is a countable intersection of internal sets. This shows that $\mathcal{S}_E$ is measurable, as it is a countable union of measurable sets.

By countable additivity of the Loeb measure, and the fact that the $\mathcal{S}_E^i$ form a nested sequence of sets, there must exist an m such that

$$\mu_{[-H,H]^d}(\mathcal{S}_E^m)\geq\alpha-\epsilon.$$

This m must now satisfy the statement, since for any k,

$$\{z\in[-H,H]^d : z+[-k,k]^d\subseteq E+[-m,m]^d\}$$

is an internal set that contains $\mathcal{S}_E^m$. □

The proposition above is false if we replace the conclusion

$$\mu_{[-H,H]^d}(\{z \in [-H,H]^d : z + [-k,k]^d \subseteq E + [-m,m]^d\}) \geq \alpha - \epsilon$$

with

$$\mu_{[-H,H]^d}(\{z \in [-H,H]^d : z + [-k,k]^d \subseteq E + [-m,m]^d\}) \geq \alpha,$$

as the example below shows (in one dimension). In this example the gaps are arbitrarily large, but only on relatively small intervals.

Example 2.6. *We define a standard sequence A by:*

$$A = \bigcup_{j=1}^{\infty} \left(\bigcup_{i=1}^{j} (i \cdot \mathbb{N}) \cap \left[2^j - 2^{j-i}, 2^j - 2^{j-i-1}\right] \right).$$

Then it is easy to see that for any infinite H, $\mu_{[1,H]}(\mathcal{S}_{{}^*A}) = 1$, but that for any given natural number m,

$$\mu_{[1,H]}(\{a \in [1,H] : [a, a+k] \subset {}^*A + [0,m]\}) \leq 1 - \frac{1}{2^m}.$$

3. Upper syndeticity and sumsets

Supose that α is a positive real number less than or equal to 1. We say that a subset A of $\mathbb{Z}^d$ is:

- *lower syndetic of level* α iff there exists a natural number $m \in \mathbb{N}$ such that for all $k \in \mathbb{N}$,
$$\underline{\mathrm{d}}(\{z \in \mathbb{Z}^d : z + [-k,k]^d \subseteq A + [-m,m]^d\}) \geq \alpha;$$
- *upper syndetic of level* α iff there exists a natural number $m \in \mathbb{N}$ such that for all $k \in \mathbb{N}$,
$$\overline{\mathrm{d}}(\{z \in \mathbb{Z}^d : z + [-k,k]^d \subseteq A + [-m,m]^d\}) \geq \alpha;$$
- *strongly upper syndetic of level* α iff for any infinite sequence $S \subseteq \mathbb{N}$, there exists $m \in \mathbb{N}$ such that for any $k \in \mathbb{N}$

$$\limsup_{n \in S} \frac{1}{(2n+1)^d} \left| \left\{ z \in [-n,n]^d : z + [-k,k]^d \subseteq A + [-m,m]^d \right\} \right| \geq \alpha.$$

In accordance with previous definitions, if any of the above holds with $m = 0$ we may replace the word "syndetic" with the word "thick." Thus a subset A of $\mathbb{Z}^d$ is:

- *lower thick of level* α iff for all $k \in \mathbb{N}$,
$$\underline{\mathrm{d}}(\{z \in \mathbb{Z}^d : z + [-k,k]^d \subseteq A\}) \geq \alpha;$$
- *upper thick of level* α iff for all $k \in \mathbb{N}$,
$$\overline{\mathrm{d}}(\{z \in \mathbb{Z}^d : z + [-k,k]^d \subseteq A\}) \geq \alpha.$$

We note that there is no need for the notion of "strongly upper thick," since it would be equivalent to that of "lower thick." We also note that lower syndeticity of level α implies strong upper syndeticity of level α, which in turn is stronger than upper syndeticity of level α. It is also trivial that a lower thick set of level α is, in particular, lower syndetic of level α; the same fact holds for upper syndeticity of level α.

Observe that replacing the upper density with the Banach density in the definition of upper syndetic of level α would make the notion trivialize, since every piecewise syndetic set would satisfy that condition with $\alpha = 1$.

Proposition 3.1. *Let $H \in {}^*\mathbb{N} \setminus \mathbb{N}$, and S be a measurable subset of $[-H,H]^d$ with the property that $\mu_{[-H,H]^d}(S + \mathbb{Z}^d) = \mu_{[-H,H]^d}(S) = \alpha > 0$. Then for each standard $k \in \mathbb{N}$*

$$\mu_{[-H,H]^d}(\{x \in \ S : x + [-k,k]^d \subseteq S\}) = \alpha.$$

Proof. Define

$$\gamma = \mu_{[-H,H]}\left(\left\{x \in S : x + [-k,k]^d \nsubseteq S\right\}\right).$$

Observe that

$$\mu_{[-H,H]^d}\left(S + \mathbb{Z}^d\right) \geq \mu_{[-H,H]}\left(S + [-k,k]^d\right) \geq \mu_{[-H,H]}\left(S\right) + \frac{\gamma}{(2k+1)^d}.$$

Here the $(2k+1)^d$ term comes from the fact that a single point in $\left(S + [-k,k]^d\right) \setminus S$ witnesses that $x + [-k,k]^d \nsubseteq S$ for at most $(2k+1)^d$ elements x of S. Since

$$\mu_{[-H,H]^d}\left(S + \mathbb{Z}^d\right) = \mu_{[-H,H]^d}\left(S\right)$$

by assumption, this implies that $\gamma = 0$. The conclusion follows. $\square$

Corollary 3.2 is straightforward but will be useful, and follows immediately from the previous result.

Corollary 3.2. *Let $H \in {}^*\mathbb{N} \setminus \mathbb{N}$, and S be a measurable subset of $[-H,H]^d$ with the property that $\mu_{[-H,H]^d}(S) = 1$. Then for each standard $k \in \mathbb{N}$*

$$\mu_{[-H,H]^d}(\{x \in \ S : x + [-k,k]^d \subseteq S\}) = 1.$$

Proposition 3.3. *Let A be a subset of $\mathbb{Z}^d$. If $\lim_{i\to\infty}(\overline{\mathrm{d}}(A^{[i]})) = \overline{\mathrm{d}}(A) = \alpha > 0$ then A is upper thick of level α.*

Proof. By Proposition 1.1 there exists an $H \in {}^*\mathbb{N} \setminus \mathbb{N}$ such that

$$\mu_{[-H,H]^d}({}^*A) = \alpha,$$

and since $\lim_{i\to\infty}(\overline{\mathrm{d}}(A^{[i]})) = \alpha$ it must be that for all $i \in \mathbb{N}$

$$\mu_{[-H,H]^d}({}^*A^{[i]}) = \alpha,$$

for if there were any finite i and any $H \in {}^*\mathbb{N} \setminus \mathbb{N}$ for which $\mu_{[-H,H]^d}({}^*A^{[i]}) > \alpha$, that would imply that $\lim_{i\to\infty}(\overline{\mathrm{d}}(A^{[i]})) > \alpha$. We now have that for all $j \in \mathbb{N}$ there exists $i_j > j$ such that

$$\alpha - 1/j \leq \frac{\left|{}^*A^{[i_j]} \cap [-H,H]^d\right|}{(2H+1)^d} \leq \alpha + 1/j.$$

By overspill there exists a J in $[1,H]$ such that i_J/H is infinitesimal and

$$\alpha - 1/J \leq \frac{\left|{}^*A^{[i_J]} \cap [-H,H]^d\right|}{(2H+1)^d} \leq \alpha + 1/J,$$

so that

$$\mu_{[-H,H]^d}({}^*A^{[i_J]}) = \alpha.$$

The set ${}^*A^{[i_J]}$ consists of a union of hypercubes of the form $(i_J)x + [0, i_J - 1]^d$. Let $N = \lfloor H/i_J \rfloor$, and

$$K = \left|\left\{x \in [-N,N]^d : (i_J)x + [0, i_J - 1]^d \subseteq {}^*A^{[i_J]}\right\}\right|.$$

Then, since i_J is infinitesimal compared to H, and every hypercube of the form $(i_J)x + [0, i_J - 1]^d$ is either completely contained in ${}^*A^{[i_J]}$ or is disjoint from it we may conclude that

$$\mathrm{st}\left(\frac{K}{(2N+1)^d}\right) = \mu_{[-H,H]^d}({}^*A^{[i_J]}) = \alpha.$$

But K is also equal to

$$\left|\left\{x \in [-N,N]^d : (i_J)x + [0, i_J - 1]^d \cap {}^*A\right\}\right| \neq \varnothing.$$

Since $\mu_{[-H,H]^d}({}^*A) = \alpha$, this implies that on almost every such cube

$$\mu_{(i_J)x+[0,i_J-1]^d}({}^*A) = 1.$$

By Corollary 3.2, for each standard $k \in \mathbb{N}$

$$\mu_{(i_J)x+[0,i_J-1]^d}(\{x \in {}^*A : x + [-k,k]^d \subseteq {}^*A\}) = 1.$$

Summing over all the N blocks that intersect ${}^*A$ now yields the desired result. □

The analogous result does not hold for lower density. There exist sets $A \subseteq \mathbb{Z}^d$ such that $\lim_{i\to\infty}(\underline{\mathrm{d}}(A^{[i]})) = \underline{\mathrm{d}}(A) = \alpha > 0$ but A is not lower thick of level β for any $\beta > 0$. See the remarks after Example 3.5 below. However the proof above can easily be adapted to show that if $\underline{\mathrm{d}}(A) = \alpha > 0$ and $\lim_{i\to\infty}(\overline{\mathrm{d}}(A^{[i]})) = \alpha$, then A is lower thick of level α.

Theorem 3.4. *Let A and B be subsets of $\mathbb{Z}^d$ with the property that $\overline{\mathrm{d}}(A) = \alpha > 0$ and $\mathrm{BD}(B) > 0$. Then $A + B$ is upper syndetic of level α.*

Proof. If $\lim_{i\to\infty}(\overline{\mathrm{d}}(A^{[i]})) = \alpha$ then by Proposition 3.3 A itself is upper thick of level α, so $A+B$ is certainly upper thick of level α.

So, it suffices to assume that there exists $i \in \mathbb{N}$ such that $\overline{\mathrm{d}}(A^{[i]}) > \alpha$. This implies that there exists $H \in {}^*\mathbb{N}\setminus\mathbb{N}$ such that

$$\mu_{[-H,H]^d}({}^*A + \mathbb{Z}^d) \geq \mu_{[-H,H]^d}({}^*A^{[i]}) > \alpha.$$

Let $\epsilon > 0$ be less than $\mu_{[-H,H]^d}({}^*A + \mathbb{Z}^d) - \alpha$. Since $\mathrm{BD}(B) > 0$ there exist arbitrarily large standard j such that for some $k \in \mathbb{Z}^d$

$$\frac{\left|B \cap (k + [-j,j]^d)\right|}{(2j+1)^d} > \mathrm{BD}(B)/2.$$

Let $\nu \in [1,H]\setminus\mathbb{N}$ be such that ν/H is infinitesimal. By Proposition 1.1 there exists $J \in {}^*\mathbb{N}\setminus\mathbb{N}$ and $k \in [-H,H]^d$ such that

$$\frac{\left|{}^*B \cap (k + [-J,J]^d)\right|}{(2J+1)^d} > \mathrm{BD}(B)/2,$$

and $|k|$ and J are less than ν. Along with Theorem 2.1, this shows that there is a point of density b of ${}^*B$ such that $\frac{|b|}{H}$ is infinitesimal. Then by Proposition 2.4 every point in $b + \mathcal{D}_{^*A} + \mathbb{Z}^d$ is a syndetic point of ${}^*(A+B)$, and since $|b|$ is infinitesimal to H, almost all elements of $b + (\mathcal{D}_{^*A} \cap [-H,H]^d)$ are in $[-H,H]^d$.

By Theorem 2.1 we know that

$$\mu_{[-H,H]^d}(\mathcal{D}_{^*A}) = \mu_{[-H,H]^d}({}^*A + \mathbb{Z}^d) > \alpha + \epsilon,$$

so that

$$\mu_{[-H,H]^d}(\mathcal{S}_{^*(A+B)}) \geq \mu_{[-H,H]^d}({}^*A + \mathbb{Z}^d) > \alpha + \epsilon.$$

By Proposition 2.5 there must exist a standard $m \in \mathbb{N}$ such that for all (standard) $k \in \mathbb{N}$

$$\mu_{[-H,H]^d}\left(\left\{z \in [-H,H]^d : z + [-k,k]^d \subseteq {}^*(A+B) + [-m,m]^d\right\}\right) \geq \alpha.$$

By the nonstandard characterization of upper asymptotic density (Proposition 1.1) we obtain the desired result. □

The theorem above is, in general, the best possible, as is shown in the example below in one dimension, with densities defined on $[1,n]$ rather than $[-n,n]$.

Example 3.5. *Let*

$$A = \bigcup_{n=1}^{\infty} [2^n, 2^n + 2^{n-1}] \text{ and } B = \bigcup_{n=1}^{\infty} [n!, n! + n].$$

then

$$\underline{\mathrm{d}}(A) = 1/2 = \underline{\mathrm{d}}(\{n : [n, n+k] \subseteq A + B + [0,m]\}))$$

for all m, and

$$\overline{\mathrm{d}}(A) = 2/3 = \overline{\mathrm{d}}(\{n : [n, n+k] \subseteq A + B + [0,m]\}))$$

for all m (note that these densities would be $1/4$ and $1/3$ if we defined the densities on $[-n, n]$ as in our general definition).

We note that if A is the set from the Example 3.5, and the set C is defined to equal $A \cap [(2n)!, (2n+1)!)$ on all $[(2n)!, (2n+1)!)$, and $(2\mathbb{N}) \cap [(2n+1)!, (2n+2)!)$ on all $[(2n+1)!, (2n+2)!)$, then C is an example of a set with the property that $\lim_{i\to\infty}(\underline{\mathrm{d}}(C^{[i]})) = \underline{\mathrm{d}}(C) > 0$ but C is not lower thick of level β for any $\beta > 0$.

In Example 3.5 where the results are sharp we note that the densities are the same for A as they are for every $A^{[j]}$ for j finite, and the conclusion holds with $m = 0$. This suggests the following slightly stronger version of the theorem above. The proof is immediate from the proof of the theorem above.

Corollary 3.6. *Let A and B be subsets of $\mathbb{Z}^d$ with the property that $\overline{\mathrm{d}}(A) = \alpha > 0$ and $\mathrm{BD}(B) > 0$. Let $\alpha' = \lim_{i\to\infty}(\overline{\mathrm{d}}(A_{[i]}))$. If $\alpha' > \alpha$ then $A + B$ is upper syndetic of level r for any $r < \alpha'$. If $\alpha = \alpha'$ then $A + B$ is upper thick of level α.*

Combining ideas from Example 2.6 and Example 3.5 it is easy to see that Corollary 3.6 cannot be improved to allow r to equal α'. The set A from Example 2.6 has $\lim_{i\to\infty}(\overline{\mathrm{d}}(A_{[i]})) = 1$, and if we add that set to the set B from Example 3.5 then $A + B$ is not upper syndetic of level 1.

4. Lower syndeticity and sumsets

In this section we focus on how the previous theorem can be improved if the set A has the stronger property of positive lower density. In the proof of Theorem 3.4 we used the fact that if $C \subseteq \mathbb{Z}^d$ is such that for some $H \in {}^*\mathbb{N} \setminus \mathbb{N}$

$$\mu_{[-H,H]^d}\left(\mathcal{S}_{^*C}\right) > \alpha,$$

then C is upper syndetic of level α. The analogous result for lower density is far from true, as Example 4.1 shows (in one dimension).

Example 4.1. *The set C constructed below has the property that almost all points in ${}^*C$ (on any infinite interval) are points of syndeticity of ${}^*C$, and $\underline{\mathrm{d}}(C) = 1/2$. However, for any m*

$$\underline{\mathrm{d}}\{n \in \mathbb{N} : n + [-2m, 2m] \subseteq A + [0, m]\} = 0.$$

Let s_i be the sequence 1,2,1,2,3,1,2,3,4,1,2,3,4,5..., and let $C \subseteq \mathbb{N}$ be such that:

$$\text{on } [i!, (i+1)!),\ n \in C \text{ iff } n \equiv \{0, 1, .., s_i - 1\} \bmod 2s_i.$$

Thus, on $[i!, (i+1)!)$, C consists of blocks of length s_i, with the blocks alternating between being completely contained in C and not intersecting C. We note that for any given m, if we let $H = (I+1)!$ and s_I be such that $2m < s_I < 3m$ then

$$\mu_{[1,H]}\left(\{n \in {}^*\mathbb{N} : n + [-2m, 2m] \subseteq {}^*C + [0, m]\}\right) = 0.$$

We also note that the only points in $^*C \setminus C$ *that are not points of syndeticity are those that are within a standard distance of an endpoint of one of the intervals of nonstandard length.*

Example 4.1 shows that we cannot use the same proof technique from Section 3 if we want to prove an analogous result with a conclusion involving lower density. These techniques do allow us to conclude strong upper syndeticity.

Theorem 4.2. *Let A and B be subsets of $\mathbb{Z}^d$ with the property that $\underline{\mathrm{d}}(A) = \alpha > 0$ and $\mathrm{BD}(B) > 0$. Then $A + B$ is strongly upper syndetic of level α.*

Proof. Let $S \subseteq \mathbb{N}$ be any sequence going to infinity. Let $H = s_I$, where $I \in {}^*\mathbb{N} \setminus \mathbb{N}$. Since $\underline{\mathrm{d}}(A) = \alpha$ we know that $\mu_{[-H,H]^d}({}^*A) \geq \alpha$. If

$$\mu_{[-H,H]^d}({}^*A + \mathbb{Z}^d) = \mu_{[-H,H]^d}({}^*A),$$

then by Proposition 3.1, for each standard $k \in \mathbb{N}$

$$\mu_{[-H,H]^d}(\{x \in {}^*A : x + [-k,k]^d \subseteq^* A\}) = \alpha,$$

and we may let $m = 0$.

If

$$\mu_{[-H,H]^d}({}^*A + \mathbb{Z}^d) > \mu_{[-H,H]^d}({}^*A),$$

then by arguments identical to those used in the proof of Theorem 3.4, there must exist $m \in \mathbb{N}$ such that for all $k \in \mathbb{N}$

$$\mu_{[-H,H]^d}\left(\left\{z \in [-H,H]^d : z + [-k,k]^d \subseteq {}^*(A+B) + [0,m]^d\right\}\right) \geq \alpha.$$

The result now follows by transfer, since for this m, any $\epsilon > 0$ and any $i, k \in \mathbb{N}$ there exists $j > i$ such that

$$\frac{1}{(2s_j+1)^d}\left|\left\{z \in [-s_j,s_j]^d : z + [-k,k]^d \subseteq A + B + [0,m]^d\right\}\right| \geq \alpha - \epsilon.$$

□

In fact it is not true that the conclusion in the theorem above can be improved to "$A + B$ is lower syndetic of level α" (see Example 4.4 below). The following theorem is the strongest conclusion we can make involving lower syndeticity.

Theorem 4.3. *Let A and B be subsets of $\mathbb{Z}^d$ with the property that $\underline{\mathrm{d}}(A) = \alpha > 0$ and $\mathrm{BD}(B) > 0$. Then for any $\epsilon > 0$, $A + B$ is lower syndetic of level $\alpha - \epsilon$.*

Proof. Without loss of generality, let $\epsilon < \alpha/2$. So we can assume that $\alpha - \epsilon > \epsilon$. Choose $m \in \mathbb{N}$ sufficiently large so that $\mathrm{BD}(B + [-m,m]^d) > 1 - \epsilon$. Let $B_m = B + [-m,m]^d$. It suffices to show that for any $H \in {}^*\mathbb{N} \setminus \mathbb{N}$ and any $k \in \mathbb{N}$,

$$\mu_{[-H,H]^d}\left(\left\{x \in [-H,H]^d : x + [-k,k]^d \subseteq {}^*(A+B) + [-m,m]^d\right\}\right) \geqslant \alpha - \epsilon.$$

We first note that by a pigeonhole argument we can prove that if $x, y, x+y \in [-H,H]^d$ are such that

$$\overline{\mathrm{d}}((^*A-x)\cap\mathbb{Z}^d)+\underline{\mathrm{d}}((^*B_m-y)\cap\mathbb{Z}^d)>1, \tag{4.1}$$

$$\text{then } (x+y+\mathbb{Z}^d)\cap[-H,H]^d \subseteq {}^*(A+B_m). \tag{4.2}$$

This is true because if x and y satisfy 4.1 then any $x' \in x+\mathbb{Z}^d$ and $y' \in y+\mathbb{Z}^d$ also satisfy this condition, so that $^*A-x'$ and $y'-{}^*B_m$ must intersect.

Now for any $n \in \mathbb{N}$, let

$$S_n = \left\{x \in {}^*A\cap[-H,H]^d : \frac{|(x+[-n,n]^d)\cap{}^*A)|}{(2n+1)^d} > \frac{\epsilon}{2}\right\} \text{ and } S=\bigcap_{N=1}^{\infty}\bigcup_{n=N}^{\infty} S_n,$$

$$T_n = \left\{x \in {}^*A\cap[-H,H]^d : \frac{|(x+[-n,n]^d)\cap{}^*A)|}{(2n+1)^d} < \frac{2\epsilon}{3}\right\} \text{ and } T=\bigcup_{N=1}^{\infty}\bigcap_{n=N}^{\infty} T_n.$$

It is now easy to verify the following **Facts**:

1) $^*A\setminus S \subseteq T$.

2) If $x \in [-H,H]^d$, then $x \in S$ implies $\overline{\mathrm{d}}((^*A-x)\cap\mathbb{Z}^d) \geqslant \epsilon/2$ and $\overline{\mathrm{d}}((^*A-x)\cap\mathbb{Z}^d) > \epsilon/2$ implies $x \in S$.

3) If $x \in {}^*A\cap[-H,H]^d$, then $x \in T$ implies $\underline{\mathrm{d}}((^*A-x)\cap\mathbb{Z}^d) \leqslant 2\epsilon/3$ and $\overline{\mathrm{d}}((^*A-x)\cap\mathbb{Z}^d) < 2\epsilon/3$ implies $x \in T$.

4) If $x,y \in {}^*A\cap[-H,H]^d$ and $x-y \in \mathbb{Z}^d$, then $x \in S$ if and only if $y \in S$ and $x \in T$ if and only if $y \in T$.

We show that $\mu_{[-H,H]^d}(T) \leqslant 2\epsilon/3$. Suppose that $\mu_{[-H,H]^d}(T)=\gamma > 2\epsilon/3$. By the Birkhoff Ergodic Theorem the asymptotic density of $T-x$ exists for almost all x (see e.g. [10] pages 23 and 24 for more details on a similar argument using this theorem). So there exists an $x \in T$ such that $d((T-x)\cap\mathbb{Z}^d) \geq \gamma$. By fact (4) above, we have that

$$(^*A-x)\cap\mathbb{Z}^d = (T-x)\cap\mathbb{Z}^d.$$

Therefore, $\underline{\mathrm{d}}((^*A-x)\cap\mathbb{Z}) = \gamma > 2\epsilon/3$, which contradicts that $x \in T$ by fact (3) above.

By fact (1) above we have that $\mu_H(S) \geqslant \alpha - 2\epsilon/3 > \alpha-\epsilon$. Let $t \in [-H,H]^d$ be such that $\|t\|/H \approx 0$ and $d(^*B_m\cap(t+\mathbb{Z}^d)) > 1-\epsilon$. By (4.1)–(4.2), we have that

$$(^*(A+B)+[-m,m]^d)\cap[-H,H]^d = (^*A+{}^*B_m)\cap[-H,H]^d \supseteq (t+S+\mathbb{Z}^d)\cap[-H,H]^d.$$

Consequently we have that for any $k \in \mathbb{N}$,

$$\mu_{[-H,H]^d}\left(\{x\in[-H,H]^d : x+[0,k]^d \subseteq {}^*(A+B)+[-m,m]^d\}\right) \geqslant \mu_H(S+\mathbb{Z}^d) \geqslant \alpha-\epsilon.$$

This completes the proof. □

Example 4.4 shows (in one dimension) that we may not replace $\alpha-\epsilon$ with α in the conclusion of the previous theorem.

Example 4.4. *Sets $A, B \subseteq \mathbb{N}$ can be constructed so that they satisfy that $\underline{\mathrm{d}}(A) = 1/2$, $\mathrm{BD}(B) \geqslant 8/9$, and for any $m \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that*

$$\underline{\mathrm{d}}(\{x \in \mathbb{N} : x + [0,k] \subseteq A + B + [0,m]\}) < \frac{1}{2}.$$

Proof. We construct A first. Let $f(n,p) = 10^{(n^2+p)^2}$. Notice that for any n and $p \leqslant n$, we have that $f(n,p) < f(n+1,0)$ and for any hyperfinite N and $p \leqslant N$, we have that $f(N,p)/(f(N,p+1) - f(N,p)) \approx 0$ and $f(N,p)/(f(N+1,0) - f(N,N)) \approx 0$.

For each $p \in \mathbb{N}$ let $r_p = \frac{10^p - 2}{2(10^p - 1)}$. Notice that $r_p < \frac{1}{2}$ when p is finite and $r_p \to \frac{1}{2}$ as $p \to \infty$. The number r_p satisfies that $r_p + \frac{1}{10^p}(1 - r_p) = \frac{1}{2}$.

For each $n \in \mathbb{N}$ and each $p = 1, 2, \ldots n-1$, let

$$C_{n,p} = C'_{n,p} \cup C''_{n,p}$$

$$\text{where } C'_{n,p} = [f(n,p), f(n,p) + r_p(f(n,p+1) - f(n,p))]$$

$$\text{and } C''_{n,p} = 10^p[f(n,p) + r_p(f(n,p+1) - f(n,p)), f(n,p+1)],$$

i.e. $C_{n,p} \subseteq [f(n,p), f(n,p+1)]$ is the union of an interval of length $r_p(f(n,p+1) - f(n,p))$ and an arithmetic progression of difference 10^p and length $\frac{1}{10^p}(1 - r_p)(f(n,p+1) - f(n,p))$. Let $C_{n,n} = 2[f(n,n), f(n+1,0)]$. Let

$$A = \bigcup_{n=1}^{\infty} \bigcup_{p=0}^{n} C_{n,p}.$$

Clearly, $\underline{\mathrm{d}}(A) = 1/2$.

Now we construct B.

For each $p \in \mathbb{N}$ let $E_p = \bigcup_{k=1}^{\infty} \left(k10^{2p} - [1, 10^p]\right)$, $D_p = \mathbb{N} \setminus E_p$, and $D = \bigcap_{p=1}^{\infty} D_p$. Let $F_n = D \cap [0, 10^{2n} - 1]$. We list the following **Facts**:

1) Every interval of length 10^{2p} contains a gap of D with length at least 10^p.

2) Also $d(D) \geqslant 1 - \sum_{p=1}^{\infty} \frac{1}{10^p} = 8/9$.

3) $F_n = D_n \cap [0, 10^{2n} - 1]$.

4) For any $p' \leqslant p$

$$k10^{2p} + D_{p'} \cap [0, 10^{2p} - 1] = D_{p'} \cap [k10^{2p}, (k+1)10^{2p} - 1].$$

5) For any $p' < p'' \leqslant p$,

$$k10^{2p} + 10^{2p''} + F_{p'} \subseteq 10^{2p}\mathbb{N} + F_p.$$

6) For any $n \geqslant p$,

$$[k10^{2p}, (k+1)10^{2p} - 1] \cap D_n \subseteq k10^{2p} + F_p.$$

7) $10^{2p}\mathbb{N} + D = 10^{2p}\mathbb{N} + F_p$.

Let

$$B = \bigcup_{n=2}^{\infty} (f(n,0) + F_n).$$

Clearly, $BD(B) \geqslant d(D) \geqslant 8/9$. For each hyperfinite integer N and $1 \leqslant p < N$, let $u = \max {}^*B \cap [0, f(N,p)]$. We have that $u/f(N,p) \approx 0$. The set B is a union of F_n's translated by rapidly increasing powers of 10. It is important to observe that

$$10^{2n}\mathbb{N} + B \subseteq 10^{2n}\mathbb{N} + F_n.$$

This is true because by fact (5), we have that (a) if $f(n',0) \geq 10^n$, then

$$k10^{2n} + f(n',0) + F_{n'} = k'^{2n} + F_{n'} \subseteq 10^{2n}\mathbb{N} + F_n$$

and (b) if $f(n',0) = 10^{(n')^4} < 10^{2n}$, then

$$k10^{2n} + f(n',0) + F_{n'} = k'10^{(n')^4} + F_{n'} = k''^{2(n'+1)} + F_{n'} \subseteq 10^{2n}\mathbb{N} + F_n.$$

Now we show that the sets A and B are what we want.

Given any $m \in \mathbb{N}$, choose a $p \in \mathbb{N}$ sufficiently large so that $10^p > 2m$. Let $H = f(N, 2p+1)$. Then $(f(N,2p+1) - f(N,2p))/H \approx 1$. Let $u = \max B \cap [1, H]$. We also have $u/H \approx 0$. So

$$\begin{aligned}&({}^*A + {}^*B + [0,m]) \cap [0, f(N,2p+1)]\\ &\subseteq [0, r_{2p}(f(N,2p+1) - f(N,2p)) + u + m] \cup (C''_{N,2p} + {}^*B \cap [0,u] + [0,m]).\end{aligned}$$

Note that $(C''_{N,2p} + {}^*B \cap [0,u]) \cap [1,H] \subseteq (C''_{N,2p} + F_{2p}) \cap [1,H]$ and every interval of length 10^{2p} contains a gap of length 10^p in $(C''_{N,2p} + F_{2p}) \cap [1,H]$. Since $10^p > 2m$, every interval of length 10^{2p} in $[f(N,p) + r_p(f(N,p+1) - f(N,p)) + u + m, f(N,p+1)]$ is not entirely in ${}^*(A+B) + [0,m]$. So we can choose $k = 10^{2p}$ so that

$$\begin{aligned}&\{x \in [1,H] : x + [0,k] \subseteq {}^*(A+B) + [0,m]\}\\ &\subseteq [1, f(N,2p) + r_{2p}(f(N,2p+1) - f(N,2p)) + u].\end{aligned}$$

Hence

$$\begin{aligned}&\mu_H(\{x \in [1,H] : x + [0,k] \subseteq {}^*(A+B) + [0,m]\})\\ &\approx \frac{1}{H}(f(N,2p) + r_{2p}(f(N,2p+1) - f(N,2p)) + u) \approx r_{2p} < \frac{1}{2}.\end{aligned}$$

□

5. Syndeticity for the sum of two sets of positive lower density

In this section we focus only on the dimension 1 case, where the results from Section 4 can be improved under the assumption that both sets have positive lower density. The results use Mann's Theorem about the additivity of Schnirelmann density [17] and thus do not generalize to n dimensions in a straightforward way. For the remainder of the section the dimension is 1 and the density functions are defined on intervals of natural numbers starting at 1, as in the classical setting.

Mann's theorem asserts that if A and B are subsets of $\mathbb{N}$ such that $\sigma(A) = \alpha$ and $\sigma(B) = \beta$, then

$$\sigma((A \cup \{0\}) + (B \cup \{0\})) \geq \min\{\alpha + \beta, 1\}.$$

This guarantees that for any n

$$\frac{|((A \cup \{0\}) + (B \cup \{0\})) \cap [1, n]|}{n} \geq \min\{\alpha + \beta, 1\}.$$

So the result can at once be thought of as pertaining to either infinite sets or finite sets of natural numbers up to some n.

We first need the proposition below.

Proposition 5.1. *Let $A \subseteq \mathbb{N}$ be such that $\underline{\mathrm{d}}(A) = \alpha > 0$. Then for any $H \in {}^*\mathbb{N} \backslash \mathbb{N}$ and any $\epsilon > 0$ there exists an internal $E \subseteq {}^*A \cap \mathcal{D}_{^*A} \cap [1, H]$ such that $\sigma(E - e) \geq r - \epsilon$ for some $e \in [1, H]$, with $e/H < \epsilon$.*

Here by $\sigma(E - e)$ we mean the Schnirelmann density of the internal set $E - e$ on $[1, H - e]$, i.e. $\inf_{h \in H - e} \frac{|(E-e) \cap [1,h]|}{h}$.

Proof. We will write D_A for ${}^*A \cap \mathcal{D}_{^*A}$. Since D_A is Loeb measurable on any interval, and Loeb measurable sets are approximable from below by internal sets, for each $n \in \mathbb{N}$ there exists an internal set E_n such that

$$E_n \subseteq D_A \cap [(H/2^n), (H/2^{n-1})]$$

and

$$\mu_{[1,H]}(E_n) > (1 - \epsilon/4)\mu_{[1,H]}(D_A \cap [(H/2^n), (H/2^{n-1})]).$$

Let $m \in \mathbb{N}$ be such that

$$1/2^m < \epsilon \text{ and let } E = \bigcup_{n=1}^{2m} (E_n).$$

Then E is internal and $E \subseteq D_A \cap [1, H] = {}^*A \cap \mathcal{D}_{^*A} \cap [1, H]$.

By Theorem 2.1,

$$\mu_{[1,H]}(D_A \cap [1, K]) \geq \alpha(K/H) \text{ for all } \mathbb{N} < K < H.$$

Note that for any $n < m$, if $x \in [(H/2^n), (H/2^{n-1})]$ then $x \geq H/2^{m-1}$ and

$$\begin{aligned} \mu_{[1,H]}(E \cap [(H/2^n), x]) &\geq \mu_{[1,H]}(D_A \cap [(H/2^n), x]) - (\epsilon/4)(1/2^{n-1} - 1/2^n) \\ &\geq \mu_{[1,H]}(D_A \cap [(H/2^n), x]) - (\epsilon/4)(x/H), \end{aligned}$$

and that

$$\mu_{[1,H]}(E \cap [1, (H/2^n)]) > (1 - \epsilon/4)\mu_{[1,H]}(D_A \cap [1, (H/2^n)]) - 1/2^{2m}.$$

We now have:

$$\begin{aligned}\mu_{[1,H]}(E\cap[1,x]) &= \mu_{[1,H]}(E\cap[1,(H/2^n)]) + \mu_{[1,H]}(E\cap[(H/2^n),x]) \\ &> (1-\epsilon/4)\mu_{[1,H]}(D_A\cap[1,(H/2^n)]) - 1/2^{2m} \\ &+ \mu_{[1,H]}(D_A\cap[(H/2^n),x]) - (\epsilon/4)(x/H) \\ &\geq \mu_{[1,H]}(D_A\cap[1,x]) \\ &- \epsilon/4(\mu_{[1,H]}(D_A\cap[(H/2^n),x]) + x/H) - \epsilon/2^m \\ &\geq \alpha x/H - \epsilon/4(x/H + x/H) - (\epsilon/2)x/H \\ &\geq \alpha x/H - \epsilon x/H.\end{aligned}$$

This means that the largest element u in $[1,H]$ such that

$$|E\cap[1,u]| < (\alpha-\epsilon)u$$

is less than $(1/2^m)H$. Let $e = u+1$. We note that e must be an element of E, and that for all $e < x < H$

$$|E\cap[e+1,x]| \geq (\alpha-\epsilon)(x-e)$$

by the maximality of u. Thus $\sigma(E-e) \geq \alpha-\epsilon$ on $[1,H-e]$, and all the statements in the conclusion are satisfied. □

Theorem 5.2. *Let A and B be subsets of $\mathbb{N}$ with the property that $\underline{\mathrm{d}}(A) = \alpha > 0$, and $\underline{\mathrm{d}}(B) = \beta > 0$. Then $A+B$ is strongly upper syndetic of level $\min\{\alpha+\beta,1\}$.*

Proof. If $\alpha+\beta > 1$ then $A+B$ contains all but finitely many positive integers, hence the conclusion holds trivially with $m = 0$. So, we suppose that $\alpha+\beta \leq 1$.

Let $S \subseteq \mathbb{N}$ be any sequence going to infinity. Let $H = s_I$, where $I \in {}^*\mathbb{N}\setminus\mathbb{N}$.

By transfer (as in the proof of Theorem 4.2) it suffices to show that there exists $m \in \mathbb{N}$ such that for all $k \in \mathbb{N}$

$$\mu_{[1,H]}\left(\{z\in[1,H] : z+[-k,k] \subseteq {}^*(A+B)+[-m,m]\}\right) \geq \alpha+\beta.$$

By Proposition 5.1, for each $n \in \mathbb{N}$ there exists an internal set $E_{A,n}$ and $a_n \in E_{A,n}$ such that

$\sigma(E_{A,n}-a_n) \geq \alpha-1/n$ on $[1,H-a_n]$, $E_{A,n} \subseteq {}^*A\cap\mathcal{D}_{^*A}\cap[1,H]$ and $a_n/H < 1/n$.

Similarly, for each $n \in \mathbb{N}$ there exists $E_{B,n}$ and $b_n \in E_{B,n}$ such that

$\sigma(E_{B,n}-b_n) \geq \beta-1/n$ on $[1,H-b_n]$, $E_{B,n} \subseteq {}^*B\cap\mathcal{D}_{^*B}\cap[1,H]$ and $b_n/H < 1/n$.

By Mann's Theorem,

$$\sigma(E_{A,n}-a_n+E_{B,n}-b_n) \geq \alpha+\beta-2/n \text{ on } [1,H-(a_n+b_n)],$$

i.e.

$$\frac{|(E_{A,n}-a_n+E_{B,n}-b_n)\cap\ [1,H-(a_n+b_n)]|}{H-(a_n+b_n)} \geq \alpha+\beta-2/n.$$

This implies that

$$\mu_{[1,H]}\left(E_{A,n} - a_n + E_{B,n} - b_n\right) \cap \ [1, H - (a_n + b_n)]) \geq$$
$$(\alpha + \beta - 2/n)(1 - 2/n) \geq \alpha + \beta - 4/n,$$

Thus

$$\mu_{[1,H]}(E_{A,n} + E_{B,n}) \geq \alpha + \beta - 4/n.$$

Since each $E_{A,n}$ is in ${}^*A \cap \mathcal{D}_{{}^*A}$ and each $E_{B,n}$ is in ${}^*B \cap \mathcal{D}_{{}^*B}$, by Theorem 2.4 we know that every $E_{A,n} + E_{B,n}$ is contained in ${}^*(A+B) \cap \mathcal{S}_{{}^*(A+B)}$, so that

$$\mu_{[1,H]}({}^*(A+B) \cap \mathcal{S}_{{}^*(A+B)}) \geq \alpha + \beta.$$

Now, if

$$\mu_{[1,H]}(\mathcal{S}_{{}^*(A+B)}) > \alpha + \beta$$

then the result follows by Proposition 2.5. If, on the other hand,

$$\mu_{[1,H]}(\mathcal{S}_{{}^*(A+B)}) = \alpha + \beta$$

then, since $\mathcal{S}_{{}^*(A+B)} = \mathcal{S}_{{}^*(A+B)} + \mathbb{Z}$, it must be that

$$\alpha + \beta \leq \mu_{[1,H]}({}^*(A+B) \cap \mathcal{S}_{{}^*(A+B)}) \leq \mu_{[1,H]}(({}^*(A+B) \cap \mathcal{S}_{{}^*(A+B)}) + \mathbb{Z})$$
$$\leq \mu_{[1,H]}(\mathcal{S}_{{}^*(A+B)} + \mathbb{Z}) = \mu_{[1,H]}(\mathcal{S}_{{}^*(A+B)}) = \alpha + \beta$$

Thus, the set ${}^*(A+B) \cap \mathcal{S}_{{}^*(A+B)}$ satisfies the hypotheses of the set S in Proposition 3.1 (in one dimension). This implies that for each standard $k \in \mathbb{N}$

$$\mu_{[1,H]}(\{x \in \ {}^*(A+B) : x + [-k,k] \subseteq {}^*(A+B)\} = \alpha + \beta,$$

and the result follows with $m = 0$. □

Question: Under the same hypotheses as in the theorem above, can we conclude that for any $\epsilon > 0$ the sumset $A + B$ is lower syndetic of level $\min\{\alpha + \beta - \epsilon, 1\}$?

Currently the strongest conclusion that can be made involving lower density is the result below.

Theorem 5.3. *Let A and B be subsets of $\mathbb{N}$ with the property that $\underline{\mathrm{d}}(A) = \alpha > 0$, and $\underline{\mathrm{d}}(B) = \beta > 0$. Then for any $\epsilon > 0$ and any increasing function $f : \mathbb{N} \to \mathbb{N}$, there exists $m_f \in \mathbb{N}$ such that*

$$\underline{\mathrm{d}}(\{n \in \mathbb{N} : \ \exists\ m < m_f,\ n+[-f(m), f(m)] \subseteq A+B+[-m,m]\}) \geq \min\{\alpha+\beta-\epsilon, 1\}.$$

We note that here m_f depends only on the function, but that m may depend on n.

Proof. As before, if $\alpha + \beta > 1$ the result is immediate, so we assume that $\alpha + \beta \leq 1$ and suppose, for the sake of contradiction, that for some $\epsilon > 0$ no such m_f exists. Then there exists $r < \alpha + \beta$ such that for all $m_0 \in \mathbb{N}$ there exist arbitrarily large $n \in \mathbb{N}$ such that for all $m < m_0$

$$|\{z \in [1,n] : z + [-f(m), f(m)] \subseteq A + B + [-m,m]\}| < rn.$$

By overspill there exist $M, H \in {}^*\mathbb{N} \setminus \mathbb{N}$ such that for all $m < M$

$$|\{z \in [1,H] : z + [-f(m), f(m)] \subseteq {}^*(A+B) + [-m,m]\}| < rH,$$

so that

$$\mu_{[1,H]}\left(\{z \in [1,H] : z + [-f(m), f(m)] \subseteq {}^*(A+B) + [-m,m]\}\right) \leq r.$$

But, as in the proof of the previous theorem, we know that for any fixed $H \in {}^*\mathbb{N} \setminus \mathbb{N}$ there exists $m \in \mathbb{N}$ such that for all $k \in \mathbb{N}$

$$\mu_{[1,H]}(\{z \in [1,H] : z + [-k,k] \subseteq {}^*(A+B) + [-m,m]\}) \geq \alpha + \beta,$$

and this contradiction completes the proof. □

6. A Lebesgue density theorem for nonstandard cuts

The classical Lebesgue Density Theorem for $\mathbb{R}^d$ says that if E is a Lebesgue measurable set in $\mathbb{R}^d$ then almost every point in E is a point of density of E, i.e. almost every $x \in E$ has the property that

$$\lim_{\epsilon \to 0} \frac{\lambda(E \cap (x + (-\epsilon, \epsilon)^d))}{(2\epsilon)^d} = 1.$$

The goal of this section is to prove an analogue of the Lebesgue Density Theorem for measures induced by arbitrary cuts in ${}^*\mathbb{N}$. Let $H \in {}^*\mathbb{N} \setminus \mathbb{N}$. A *cut* U in $[1,H]$ is an initial segment of $[1,H]$ that is closed under addition. Cuts in this context were introduced in [13], and some of the topological properties of the quotient space $[1,H]$ under the equivalence relation $x \equiv y$ iff $|x-y| \in U$ were explored.

For a given cut U in $[1,H]$, we let $\mathcal{U} = (-U) \cup \{0\} \cup (U)$. A $U-$*monad of* $[-H,H]^d$ is a set of the form $x + \mathcal{U}^d$, where $x \in [-H,H]^d$ and $x + \mathcal{U}^d \subseteq [-H,H]^d$. The main result in this section is really about the behavior of Loeb measure on the space of monads of various cuts, i.e. the quotient space under the projection that sends x to $x + \mathcal{U}^d$. For any $\mathbb{N} < K \leq H$ there is a natural cut of all elements infinitesimal to K, given by $U_K = \bigcap_{i=1}^{\infty} [1, K/i]$. Loeb measure on the quotient space of $[-K,K]^d$ for the cut U_K is isomorphic to Lebesgue Measure on $[-1,1]^d$ via the measure-preserving mapping that sends $x + \mathcal{U}^d$ to $\mathrm{st}(x/K)$. So, the fact that the Lebesgue Density Theorem holds for such cuts is immediate from the fact that the result holds for Lebesgue measure. Previous standard results were obtained by using the density theorem in the space of monads of such U_K in [14], [15] and [16]. In this section we show that there is an analogous density theorem for every cut and in every finite dimension. The standard results in this paper are based on the density theorem in the case where that $U = \mathbb{N}$.

We begin with a standard combinatorial lemma.

Lemma 6.1. *Suppose that* $m \in \mathbb{N}$ *and* $(T_i)_{i<n}$ *is a collection of subsets of a finite set* X *such that for every* $x \in X$

$$1 \leq \sum_{i<n} \chi_{T_i}(x) \leq m$$

where χ_{T_i} *denotes the characteristic function of* T_i*. If* $t \in (0,1)$ *and* $E \subset X$ *is such that*

$$\frac{|T_i \cap E|}{|T_i|} \leq t$$

for every $i < n$*, then*

$$\frac{|E|}{|X|} \leq \frac{mt}{1+(m-1)\,t}.$$

Proof. We may assume that each $x \in E$ is in only one of the T_i and that each $x \in X \setminus E$ is in m of the T_i since removing elements of E from all but one of the T_i or adding elements of $X \setminus E$ to any of the T_i (if that element is in fewer than m of them) maintains the hypotheses without changing the conclusion. Then

$$|E| \leq t \sum_{i<n} |T_i| = t(m(|X| - |E|) + |E|) = t(m\,|X| - (m-1)\,|E|)$$

so that

$$|E|\,(1+(m-1)t) \leq tm\,|X|$$

which yields the desired result. □

If E is an internal subset of ${}^*\mathbb{Z}^d$ and $x \in {}^*\mathbb{Z}^d$ define

$$d_E(x) := \liminf_{\nu > U} \mu_{x+[-\nu,\nu]^d}\left(\left(E+\mathcal{U}^d\right) \cap \left(x+[-\nu,\nu]^d\right)\right),$$

where $\liminf_{\nu>U}$ means $\sup_{\xi>U} \inf_{U<\nu<\xi}$. Observe that if $x \in {}^*\mathbb{Z}^d$ and $y \in U^d$ then

$$d_E\,(x+y) = d_E\,(x-y) = d_E(x).$$

The proof of the next theorem is based on the proof of the Lebesgue Density Theorem given in [6].

Theorem 6.2. *Let* $H \in {}^*\mathbb{N} \setminus \mathbb{N}$ *and* E *be an internal subset of* $[-H,H]^d$*. Then*

$$\mu_{[-H,H]^d}\left(\left\{x \in E+\mathcal{U}^d : d_E(x) < 1\right\}\right) = 0.$$

Proof. We will write simply μ for $\mu_{[-H,H]^d}$. Until we are able to show that the outer measure of $\left\{x \in E+\mathcal{U}^d : d_E(x) < 1\right\}$ is 0, it is not clear that the set is measurable. To show this, we fix $t \in (0,1)$, and prove that the set

$$R = \left\{x \in E+\mathcal{U}^d : d_E(x) < t\right\}$$

has outer measure 0. For any $\epsilon > 0$ we may pick an internal subset D of $[-H,H]^d$ containing R such that

$$\mu\,(D) \leq \mu^*\,(R) + \epsilon.$$

We will show that

$$\mu\,(D) \leq \frac{\epsilon}{1-\frac{4^d t}{(4^d-1)t+1}},$$

which can be made arbitrarily small by making ϵ small. This yields the desired result since $R \subseteq D$.

Define

$$R_+ = \left\{ x \in E : \exists z \in [1,H], x + [-z,z]^d \subset D \quad and \quad \frac{\left|E \cap (x + [-z,z]^d)\right|}{(2z+1)^2} < t \right\}.$$

Observe that $R \cap E \subset R_+ \subset E$ and R_+ is internal. We can now cover every point x in D by open hypercubes of the form $(x + (-y,y)^d)$, such that

$$\frac{\left|R_+ \cap (x + (-y,y)^d)\right|}{(2y+1)^2} \leq t,$$

by letting $y = z + \frac{1}{2}$ if $x \in R_+$ and $y = \frac{1}{2}$ if $x \in D \setminus R_+$. By the Besicovitch Covering Theorem (see, for example, [12, page 483]) there exists a hyperfinite sequence $(S_i)_{i\in I}$ of these hypercubes such that for every $x \in D$

$$1 \leq \sum_{i \in I} \chi_{S_i}(x) \leq 4^d.$$

An application of Lemma 6.1 shows that

$$\frac{|R_+|}{|D|} \leq \frac{4^d t}{(4^d-1)t+1}.$$

It follows that

$$\begin{aligned} \mu(D) &\leq \mu(R_+) + \epsilon \\ &\leq \frac{4^d t}{(4^d-1)t+1}\mu(D) + \epsilon \end{aligned}$$

and hence

$$\mu(D) \leq \frac{\epsilon}{1 - \frac{4^d t}{(4^d-1)t+1}}$$

as desired, showing that the outer measure is 0.

By the completeness of the Loeb measure we obtain the desired result. □

Let U be a cut in $[1,H]$. We say that a subset S of $[-H,H]^d$ is *U-hereditarily measurable* iff for every $x \in [-H,H]^d$ and every $U<\nu<H$,

$$(S + \mathcal{U}^d) \cap (x + [-\nu,\nu]^d) \text{ is } \mu_{x+[-\nu,\nu]^d} - \text{measurable.}$$

For $x \in [-H,H]^d$ and $S \subseteq [-H,H]^d$ $U-$hereditarily measurable we define:

$$d_S^U(x) = \liminf_{\nu > U} \mu_{x+[-\nu,\nu]^d}\left((S + \mathcal{U}^d) \cap (x + [-\nu,\nu]^d)\right).$$

We note that since S is hereditarily measurable d_S^U is well-defined. If $U = \mathbb{N}$ and S is internal this definition agrees with the definition given in Section 2. Equivalently, we adopt the convention that if $U = \mathbb{N}$ we simply write $d_S(x)$ for $d_S^{\mathbb{N}}(x)$. As in Section 2 we say that x is a point of density of S iff

$d_S^U(x) = 1$, and we write $\mathcal{D}_S^U$ for the set of all points of density of S with respect to the cut U.

We say that a cut U has *countable cofinality* iff there exists an increasing sequence $x_n \in {}^*\mathbb{N}$ such that $\bigcup_{n\in\mathbb{N}}[1, x_n] = U$, and that U has *countable coinitiality* iff there exists a decreasing sequence $x_n \in {}^*\mathbb{N}$ such that $\bigcap_{n\in\mathbb{N}}[1, x_n] = U$.

Proposition 6.3. *For any cut U in $[1, H]$, every internal set contained in $[-H, H]^d$ is $U-$hereditarily measurable.*

Proof. Since any internal set intersected with any $x + [-\nu, \nu]^d$ is internal, it suffices to show that for any internal set E, $E+\mathcal{U}^d$ is $\mu_{x+[-\nu,\nu]^d}-$measurable. We will simply write μ for $\mu_{x+[-\nu,\nu]^d}$. If U has countable cofinality then $E + \mathcal{U}^d$ is a countable union of internal sets of the form $E \pm [1, x_n]^d$ and so is measurable. If U has countable coinitiality then $E + \mathcal{U}^d$ is a countable intersection of internal sets of the form $E\pm[1, x_n]^d$ and so is measurable. So, we assume that U has neither countable coinitiality nor countable cofinality. Let

$$\gamma = \inf\left\{\mu\left(E + [-K, K]^d\right) : K > U\right\}$$

and

$$\delta = \sup\left\{\mu\left(E + [-K, K]^d\right) : K < U\right\},$$

It suffices to show that $\gamma = \delta$. Assume the contrary, that $\gamma > \delta$.

Let $K_n > U$ be decreasing such that

$$\mu(E + [-K_n, K_n]^d) < \gamma + 1/n$$

for all $n \in \mathbb{N}$. Since the coinitiality of U is uncountable, there exists $K' > U$ such that

$$\text{for any } K \le K' \text{ and } K > U,\ \mu(E + [-K, K]^d) = \gamma.$$

Symmetrically, we can find a $K'' < U$ such that

$$\text{for any } K \ge K'' \text{ and } K < U,\ \mu(E + [-K, K]^d) = \delta.$$

Let $\eta = \frac{1}{2}(\gamma + \delta)$, and let

$$X = \left\{K \in [K'', K'] : \left|E + [-K, K]^d\right| / (2\nu + 1)^d \le \eta\right\}.$$

Then X is internal and $U \cap [K'', K'] \subseteq X$. So, $X \cap ([K'', K'] \setminus U)$ is nonempty. Let $K \le K'$ and $K > U$ be such that

$$\mu\left(E + [-K, K]^d\right) \approx \left|E + [-K, K]^d\right| / (2\nu + 1)^d \le \eta.$$

This contradicts the fact that $\gamma > \eta$. □

Proposition 6.4. *Let $H \in {}^*\mathbb{N}\setminus\mathbb{N}$, U be a cut in $[1, H]$, and $S \subseteq [-H, H]^d$ be hereditarily measurable. Then the set*

$$\left\{x \in S + \mathcal{U}^d : d_S(x) < 1\right\}$$

has Loeb measure zero relative to $[-H, H]^d$.

Proof. Fix an $\epsilon > 0$. Since $S + \mathcal{U}^d$ is measurable there exists $E \subseteq S + \mathcal{U}^d$ such that E is internal and $\mu_{[-H,H]^d}(S \setminus E) < \epsilon$. Then

$$\left\{x \in S + \mathcal{U}^d : d_S(x) < 1\right\} \subseteq \left\{x \in E + \mathcal{U}^d : d_E(x) < 1\right\} \cup (S + \mathcal{U}^d \setminus E).$$

It follows that the outer measure of

$$\left\{x \in S + \mathcal{U}^d : d_S(x) < 1\right\}$$

is at most ϵ. Since ϵ is arbitrary, the outer measure is 0, and the result follows by the completeness of the Loeb measure. □

Corollary 6.5 is a generalization of Theorem 2.1.

Corollary 6.5. *If E is an internal subset of $[-H, H]^d$ and U is a cut in $[1, H]$ then $\mathcal{D}_E^U$ is $\mu_{[-H,H]^d}$-measurable, and $\mu_{[-H,H]^d}(\mathcal{D}_E^U) = \mu_{[-H,H]^d}(E + \mathcal{U}^d)$.*

Proof. $\mathcal{D}_E^U = (E + \mathcal{U}^d) \setminus \left\{x \in E + \mathcal{U}^d : d_E(x) < 1\right\}$, and the conclusion follows. □

It would be interesting to know if the results of Section 3 and Section 4 generalize to more general amenable groups.

It would also be interesting to know if the density theorem in the space of monads of cuts other than $U = \mathbb{N}$ or some U_K can be used to obtain new standard results.

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