The Lusztig-Macdonald-Wall polynomial conjectures and q-difference equations

Kathrin Bringmann, Karl Mahlburg, and Antun Milas

In honor of Jim Lepowsky and Robert Wilson

ABSTRACT. Lusztig, Macdonald, and Wall studied certain polynomials that arose in the study of the representations of finite classical groups. We revisit subsequent work of Andrews in which he proved conjectural formulas for the infinite limits of these polynomials. Our main result provides a new proof of these conjectures by considering the generating function of the polynomials, and then calculating a hypergeometric q-series solution to the associated q-difference equation. We also connect these polynomials with the basic representations of the affine Lie algebra $\widehat{\mathfrak{sl}}_2$.

1. Introduction and statement of results

In [**1, 2**], Andrews proved identities for certain polynomials arising from the representation theory of finite classical groups. In particular, Wall gave generating functions for the numbers of conjugacy classes in [**10**], and subsequently Lusztig [9], and independently Macdonald (see p.2 in [1]), conjectured closed forms for these polynomials in certain limiting cases.

The polynomials are denoted by $\chi_n = \chi_n(a, b, q)$ (see (1.1) and (1.2) in [**2**]), and are defined by $\chi_{-1} = a$, $\chi_0 = b$, and for $n \in \mathbb{N}_0$ through the recurrences

$$\chi_{2n+1} = \chi_{2n} + q^{2n+1}\chi_{2n-1} \tag{1.1}$$

$$\chi_{2n+2} = \chi_{2n+1} + q^{n+1}\left(1 + q^{n+1}\right)\left(\chi_{2n+1} + \left(1 - q^{2n+1}\right)\chi_{2n-1}\right). \tag{1.2}$$

2000 *Mathematics Subject Classification.* 05A15, 05A30, 11P84, 20G40.

Key words and phrases. Lusztig-Macdonald-Wall polynomials; q-difference equations; theta functions.

The research of the first author was supported by the Alfried Krupp Prize for Young University Teachers of the Krupp Foundation and the research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP/2007-2013) / ERC Grant agreement n. 335220 - AQSER. The second author was supported by NSF Grant DMS-1201435. The third author was supported by a Simons Collaboration Grant and support from the Max Planck Institute of Mathematics, Bonn.

Andrews' main results, which proved the conjectures of Macdonald and Lusztig, concerned the limiting series given by these polynomials. Let

$$\chi(a,b,q) := \lim_{n\to\infty} \chi_n(a,b,q),$$

and refer to Section 2.1 for notation.

THEOREM ([**1**], Theorems 1 and 2). *The limiting polynomials evaluate to*

$$\chi(1,1,q) = \frac{1}{(q;q)_\infty} \sum_{j\in\mathbb{Z}} q^{j^2}, \tag{1.3}$$

$$\chi(0,1,q) = \frac{1}{(q;q)_\infty} \sum_{j\geq 0} q^{j(j+1)}. \tag{1.4}$$

Andrews derived these formulas by evaluating a certain double-series that is intimately related to the matrix system associated to (1.1) and (1.2).

In this note we use the theory of q-difference equations to prove finite formulas for the Lusztig-Macdonald-Wall polynomials.

THEOREM 1.1. *If $N \in \mathbb{N}_0$, then we have*

$$\frac{\chi_{2N-1}(a,b,q)}{(q;q^2)_N} = \frac{aq^{\frac{N(N+1)}{2}}}{(q;q)_N} + \sum_{m=0}^{N-1} \frac{q^{\frac{m(m+1)}{2}}(-q;q)_{N-1-m}}{(q;q)_m\,(q;q^2)_{N-m}} \left(\sum_{j=0}^{N-1-m} \frac{(-1)^{j+1}q^j\left(q;q^2\right)_j}{(q;q)_{j+1}(-q;q)_j}\left(Aq - B\left(1-q^{j+1}\right)\right) + 2a\right),$$

where $A := a(1-q^{-1})$ and $B := -3a+b$.

REMARK. The two specializations (1.3) and (1.4) are sufficient to solve for the polynomials with any initial conditions, as the linearity of the recurrence implies that

$$\chi_n(a,b) = a\chi_n(1,1) + (b-a)\,\chi_n(0,1).$$

Theorem 1.1 is of greatest interest because it leads to a new proof of the limiting formulas, as stated in the following result.

COROLLARY 1.2. *The limits* (1.3) *and* (1.4) *are true.*

REMARK. Note that (1.1) and (1.2) imply that the limit of χ_n as $n\to\infty$ may be evaluated by considering only the subsequence of odd indices.

In fact, Andrews also subsequently proved finite evaluations for the Lusztig-Macdonald-Wall polynomials in a particularly compact form, namely (Theorem 1 in [**2**])

$$\chi_{2N-1}(1,1,q) = \sum_{j\in\mathbb{Z}} \begin{bmatrix} 2N \\ N+2j \end{bmatrix}_q q^{j^2}, \tag{1.5}$$

$$\chi_{2N-1}(0,1,q) = \sum_{j\in\mathbb{Z}} \begin{bmatrix} 2N \\ N-1-2j \end{bmatrix}_q q^{j(j+1)}.$$

There are corresponding formulas for even indices, and they were proven in [**2**] by an inductive argument showing that they also satisfy the recurrences (1.1) and (1.2). In contrast, our proof proceeds by directly solving a q-difference equation using the theory of hypergeometric q-series, and thus does not require any additional calculations or prior guesses. The technical details of our work follow in Section 2, where we review classical results from the theory of hypergeometric q-series, and then apply these techniques to evaluate the generating function for the Lusztig-Macdonald-Wall polynomials, thereby proving Theorem 1.1 and Corollary 1.2.

It is interesting to note that the right hand-sides of the limiting formulas of the Lusztig-Macdonald-Wall polynomials are (essentially) characters of certain $\widehat{\mathfrak{sl}}_2$-modules. This connection with representation theory is explored in the last section of the paper. In particular, we construct monomial bases of certain subspaces of the basic module whose characters agree with $\chi_{2N-1}(1,1,q)$ (see Theorem 3.1). We also speculate that this connection may lead to representation theoretical interpretation of the recurrences (1.1-1.2) (see Remark 3).

2. Hypergeometric q-series and proofs of results

2.1. Hypergeometric q-series. We use the theory of hypergeometric q-series in order to solve the q-difference equation associated to the Lusztig-Macdonald-Wall polynomials. We review the standard notation and definitions, and then state several identities and evaluations that we use in the sequel.

If $a \in \mathbb{C}$ and $|q| < 1$, then the (rising) *q-factorial* is defined by $(a;q)_\infty := \prod_{n\geq 1}\left(1-aq^{n-1}\right)$. Furthermore, if k is an integer, the finite q-factorials are given by

$$(a;q)_k := \frac{(a;q)_\infty}{(aq^k;q)_\infty}.$$

If $m, n \in \mathbb{Z}$ with $n \geq 0$, then the *q-binomial coefficient* is defined by

$$\begin{bmatrix} n \\ m \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_m(q;q)_{n-m}}.$$

The first classical identity that we use is the Jacobi Triple Product ((2.2.10) in [**3**]), which holds for any $z \in \mathbb{C}$ and $|q| < 1$:

$$\sum_{n\in\mathbb{Z}} z^n q^{n^2} = \left(-zq;q^2\right)_\infty \left(-z^{-1}q;q^2\right)_\infty \left(q^2;q^2\right)_\infty. \tag{2.1}$$

We also use two identities due to Euler ((2.2.5) and (2.2.6) in [**3**]):

$$\frac{1}{(z;q)_\infty} = \sum_{n\geq 0}\frac{z^n}{(q;q)_n}, \tag{2.2}$$

$$(-z;q)_\infty = \sum_{n\geq 0}\frac{z^n q^{\frac{n(n-1)}{2}}}{(q;q)_n}. \tag{2.3}$$

We also recall Ramanujan's ${}_1\psi_1$-summation formula (equation (5.2.1) in [**6**]), which is valid for $|\frac{b}{a}| < |z| < 1$:

$$\sum_{m\in\mathbb{Z}} \frac{(a;q)_m}{(b;q)_m} z^m = \frac{(q;q)_\infty \left(\frac{b}{a};q\right)_\infty (az;q)_\infty \left(\frac{q}{az};q\right)_\infty}{(b;q)_\infty \left(\frac{q}{a};q\right)_\infty (z;q)_\infty \left(\frac{b}{az};q\right)_\infty}. \tag{2.4}$$

2.2. Proof of Theorem 1.1. In this section we construct a generating function for the polynomials. We work with generating series for the recurrences (1.1) and (1.2), reducing them to a single q-difference equation. We then find a hypergeometric q-series solution to the q-difference equation in order to recover our formulas.

Inserting (1.2) into (1.1) gives a recurrence for the odd-indexed polynomials, namely

$$\begin{aligned}\chi_{2n+1} &= \chi_{2n-1} + q^n\left(1+q^n\right)\left(\chi_{2n-1} + \left(1-q^{2n-1}\right)\chi_{2n-3}\right) + q^{2n+1}\chi_{2n-1}\\ &= \left(1+q^n+q^{2n}+q^{2n+1}\right)\chi_{2n-1} + q^n\left(1+q^n\right)\left(1-q^{2n-1}\right)\chi_{2n-3}.\end{aligned}$$

For convenience we adopt the shorter notation $Y_n := \chi_{2n-1}$, so that

$$Y_{n+1} = \left(1+q^n+q^{2n}+q^{2n+1}\right)Y_n + q^n\left(1+q^n\right)\left(1-q^{2n-1}\right)Y_{n-1}.$$

The generating function is easier to work with if we re-normalize the recurrence by setting $\gamma_n := \frac{Y_n}{(q;q^2)_n}$. The initial values are $\gamma_0 = a$ and $\gamma_1 = \frac{b+aq}{1-q}$, and the recurrence then becomes

$$\gamma_{n+1} = \left(1+q^n+q^{2n}+q^{2n+1}\right)\frac{\gamma_n}{1-q^{2n+1}} + q^n\left(1+q^n\right)\frac{\gamma_{n-1}}{1-q^{2n+1}},$$

which, after rewriting and re-indexing, is equivalent to

$$\left(1-q^{2n-1}\right)\gamma_n = \left(1+q^{n-1}+q^{2n-2}+q^{2n-1}\right)\gamma_{n-1} + q^{n-1}\left(1+q^{n-1}\right)\gamma_{n-2}. \tag{2.5}$$

For $\ell \geq 0$, define the shifted generating function $F^{(\ell)}(z) := \sum_{n\geq \ell}\gamma_n z^n$, and let $F(z) := F^{(0)}(z)$. Multiplying (2.5) by z^n and summing over $n \geq 2$ gives

$$\begin{aligned}&F^{(2)}(z) - q^{-1}F^{(2)}\left(zq^2\right)\\ =&z\left(F^{(1)}(z) + F^{(1)}(zq) + F^{(1)}\left(zq^2\right)(1+q)\right) + z^2\left(qF(zq) + q^2F\left(zq^2\right)\right).\end{aligned}$$

After adding the missing initial terms, we obtain the nonhomogeneous q-difference equation

$$(1-z)F(z) = z(1+zq)F(zq) + q^{-1}(1+zq)\left(1+zq^2\right)F\left(zq^2\right) + A + Bz, \tag{2.6}$$

where A and B are defined in Theorem 1.1

The q-difference equation is further simplified by setting $H(z) := \frac{F(z)}{(-zq;q)_\infty}$, so that (2.6) becomes

$$(1-z)H(z) = zH(zq) + q^{-1}H\left(zq^2\right) + \frac{A+Bz}{(-zq;q)_\infty}. \tag{2.7}$$

Note that the coefficients of the terms involving H are linear expressions in z (this is the "homogeneous part" of the q-difference equation), which means that the hypergeometric q-series solution can be found directly through an inductive argument. In particular, if we denote the series coefficients by $H(z) =: \sum_{n\geq 0} \delta_n z^n$, then (2.7) implies

$$\sum_{n\geq 0}(1-z)\delta_n z^n = \sum_{n\geq 0} \delta_n q^n z^{n+1} + \sum_{n\geq 0} q^{2n-1}\delta_n z^n + (A+Bz)\sum_{n\geq 0}\frac{(-z)^n q^n}{(q;q)_n}, \tag{2.8}$$

where the final series expansion follows from (2.2).

Isolating the coefficient of z^n in (2.8) then implies that for $n \geq 1$

$$\delta_n = \frac{1+q^{n-1}}{1-q^{2n-1}}\delta_{n-1} + \frac{(-1)^n q^{n-1}}{(1-q^{2n-1})(q;q)_n}\left(Aq - B\left(1-q^n\right)\right),$$

with initial term $\delta_0 = \gamma_0 = a$. An inductive argument then gives the following formula for all $n \in \mathbb{N}_0$:

$$\delta_n = \sum_{j=0}^{n-1} \frac{(-1)^{j+1} q^j (-q;q)_{n-1}\left(q;q^2\right)_j\left(Aq - B\left(1-q^{j+1}\right)\right)}{(q;q)_{j+1}(-q;q)_j\left(q;q^2\right)_n} + \frac{(-1;q)_n}{\left(q;q^2\right)_n}a.$$

Thus we finally have the hypergeometric solution

$$\begin{aligned} F(z) = (-zq;q)_\infty H(z) = \sum_{m\geq 0}\frac{z^m q^{\frac{m(m+1)}{2}}}{(q;q)_m} \\ \times \left[\sum_{n\geq 1}\frac{z^n(-q;q)_{n-1}}{\left(q;q^2\right)_n}\left(\sum_{j=0}^{n-1}\frac{(-1)^{j+1}q^j\left(q;q^2\right)_j\left(Aq-B\left(1-q^{j+1}\right)\right)}{(q;q)_{j+1}(-q;q)_j} + 2a\right) + a\right], \end{aligned}$$

where the summation on m follows from (2.3). This is equivalent to the statement of Theorem 1.1 under the change of summation variables $N = m + n$.

2.3. Proof of Corollary 1.2. In this section, we evaluate the infinite limit of the χ_n in order to obtain the product formulas (1.3) and (1.4).

Using the finite formula from Theorem 1.1 and continuing with the same notation as in the previous section, we calculate

$$\begin{aligned} \lim_{N\to\infty} Y_N &= \lim_{N\to\infty} \gamma_N\left(q;q^2\right)_N \\ &= (-q;q)_\infty \sum_{m\geq 0}\frac{q^{\frac{m(m+1)}{2}}}{(q;q)_m}\left(\sum_{j\geq 0}\frac{(-1)^{j+1}q^j\left(q;q^2\right)_j\left(Aq-B\left(1-q^{j+1}\right)\right)}{(q;q)_{j+1}(-q;q)_j} + 2a\right). \end{aligned}$$

The sum on m evaluates to $(-q;q)_\infty$ by (2.3) (with $z = q$), giving

$$\lim_{N\to\infty} Y_N = (-q;q)_\infty^2\left(A\sum_{j\geq 0}\frac{(-1)^{j+1}q^{j+1}\left(q;q^2\right)_j}{(q;q)_{j+1}(-q;q)_j} + B\sum_{j\geq 0}\frac{(-1)^j q^j\left(q;q^2\right)_j}{\left(q^2;q^2\right)_j} + 2a\right). \tag{2.9}$$

In order to prove the corollary statement, we now consider the two specializations. If $a = 0$ and $b = 1$, then $A = 0$ and $B = 1$. By (2.4) (with $q \mapsto q^2, a = q, b = q^2$, and $z = -q$), the sum evaluates to

$$\sum_{j\geq 0}\frac{(-1)^j q^j \left(q;q^2\right)_j}{\left(q^2;q^2\right)_j}=\sum_{j\in\mathbb{Z}}\frac{(-1)^j q^j \left(q;q^2\right)_j}{\left(q^2;q^2\right)_j}=\frac{\left(-q^2;q^2\right)_\infty}{\left(-q;q^2\right)_\infty}. \tag{2.10}$$

$$\sum_{j\in\mathbb{Z}}\frac{(-1)^j q^j \left(q;q^2\right)_j}{\left(q^2;q^2\right)_j}=\frac{\left(q^2;q^2\right)_\infty\left(q;q^2\right)_\infty\left(-q^2;q^2\right)_\infty\left(-1;q^2\right)_\infty}{\left(q^2;q^2\right)_\infty\left(q;q^2\right)_\infty\left(-q;q^2\right)_\infty\left(-1;q^2\right)_\infty}.$$

Plugging in to (2.9), we therefore have

$$\chi(0,1,q)=(-q;q)_\infty\left(-q^2;q^2\right)_\infty^2,$$

which is equivalent to (1.4) by (2.1).

If $a=b=1$, then $A=1-q^{-1}$ and $B=-2$. We must therefore also evaluate the first sum from (2.9). First, observe that

$$\begin{aligned}\sum_{j\geq 0}\frac{(-1)^{j+1}q^{j+1}\left(q;q^2\right)_j}{(q;q)_{j+1}(-q;q)_j}&=\sum_{j\geq 0}\frac{(-1)^{j+1}q^{j+1}\left(q;q^2\right)_j\left(1+q^{j+1}\right)}{\left(q^2;q^2\right)_{j+1}}\\&=\sum_{j\in\mathbb{Z}}\frac{(-1)^{j+1}q^{j+1}\left(q;q^2\right)_j\left(1+q^{j+1}\right)}{\left(q^2;q^2\right)_{j+1}}-\frac{2}{1-q^{-1}},\end{aligned}$$

where the extra term arises from $j=-1$.

After further rewriting the sum, we apply (2.4) twice, obtaining

$$\begin{aligned}\sum_{j\in\mathbb{Z}}\frac{(-1)^{j+1}q^{j+1}\left(q;q^2\right)_j\left(1+q^{j+1}\right)}{\left(q^2;q^2\right)_{j+1}}&=\frac{-q}{1-q^2}\sum_{j\in\mathbb{Z}}\frac{(-1)^{j}q^{j}\left(q;q^2\right)_j\left(1+q^{j+1}\right)}{\left(q^4;q^2\right)_{j}}\\&=-\frac{q}{1-q^2}\left(\frac{(1-q^2)(-1;q^2)_\infty}{(1-q)(-q;q^2)_\infty}+\frac{\left(-q^{-1};q^2\right)_\infty}{\left(-q^2;q^2\right)_\infty}\right)\\&=-\frac{2q(-q^2;q^2)_\infty}{(1-q)(-q;q^2)_\infty}-\frac{q(-q;q^2)_\infty}{(1-q)(-q^2;q^2)_\infty}.\end{aligned}$$

Plugging in to (2.9) and recalling (2.10), the resulting expression simplifies to

$$\chi(1,1,q)=\frac{(-q;q)_\infty^2(-q;q^2)_\infty}{(-q^2;q^2)_\infty},$$

which is similarly equivalent to (1.3) by (2.1).

As in the remark after Theorem 1.1, we immediately find the general linear formula

$$\chi\left(a,b;q\right)=\frac{\left(q^2;q^2\right)_\infty^2}{(q;q)_\infty^2}\left(a\cdot\frac{\left(q^2;q^2\right)_\infty^3}{\left(q^4;q^4\right)_\infty^2(q;q)_\infty}+(-a+b)\cdot\frac{\left(q^4;q^4\right)_\infty^2(q;q)_\infty}{\left(q^2;q^2\right)_\infty^3}\right).$$

3. The LMW polynomials and the affine Lie algebra $\widehat{\mathfrak{sl}}_2$

In this part we connect the Lusztig-Macdonald-Wall polynomials with certain representations of the affine Lie algebra

$$\widehat{\mathfrak{sl}}_2 = \mathfrak{sl}_2 \otimes \mathbb{C}\left[t, t^{-1}\right] \oplus \mathbb{C}c.$$

Here c denotes the central element and $\mathfrak{sl}_2 = \mathrm{span}\{e, f, h\}$, equipped with the standard bracket relations. Throughout we write $x(n)$ for $x \otimes t^n$, $x \in \mathfrak{sl}_2$.

It is well known that $\widehat{\mathfrak{sl}}_2$ admits two *basic* (i.e., level one integrable highest weight) irreducible representation usually denoted by $L(\Lambda_0)$ and $L(\Lambda_1)$ [**7**]. Basic representations played a fundamental role in the pioneering work of Lepowsky and Wilson on the principal realization of $\widehat{\mathfrak{sl}}_2$ [**8**]. But rather than working within principal realization here we consider closely related Frenkel-Kac and Segal's homogeneous realization [**7**]. In this setup, we can make the following identification

$$L(\Lambda_0) = S(\mathfrak{h}_{<0}) \otimes \mathbb{C}[L], \quad L(\Lambda_1) = S(\mathfrak{h}_{<0}) \otimes e^{\frac{\alpha}{2}}\mathbb{C}[L], \tag{3.1}$$

where $L = \mathbb{Z}\alpha$ is the root lattice of $\mathfrak{sl}_2$, $\mathfrak{h} = \mathbb{C}h$, and $\mathbb{C}[L] = \mathrm{span}\{e^{m\alpha}, m \in \mathbb{Z}\}$ is the group algebra of L. If we let v_i be the highest weight vectors of $L(\Lambda_i)$, then the $\mathbb{N}$-grading on $L(\Lambda_i)$ is introduced as follows: $\mathrm{wt}(v_i) = 0$, and

$$\mathrm{wt}\left(x_1(-i_1)\cdots x_k(-i_k)v_i\right) := i_1 + \cdots + i_k,$$

where $x_i \in \mathfrak{sl}_2$. By using (3.1) we obtain the following well-known character formulae [**7**]:

$$\mathrm{ch}[L(\Lambda_0)](q) = \frac{\sum_{n\in\mathbb{Z}} q^{n^2}}{(q;q)_\infty}, \tag{3.2}$$

$$\mathrm{ch}[L(\Lambda_1)](q) = \frac{2\sum_{n\geq 0} q^{n(n+1)}}{(q;q)_\infty}, \tag{3.3}$$

where $\mathrm{ch}[L(\Lambda_i)](q)$ is the q-generating series of graded dimensions of $L(\Lambda_i)$. Note that these agree with $\lim_{n\to\infty} \chi_{2n-1}(1,1,q)$ and $2\lim_{n\to\infty} \chi_{2n-1}(0,1,q)$, respectively.

Now we describe subspaces $F_n \subset L(\Lambda_0)$, $n \geq 1$, each equipped with a nice bigraded monomial basis (via standard generators of $\widehat{\mathfrak{sl}}_2$) whose character is $\chi_{2n-1}(1,1,q)$.

Theorem 3.1. *(LMW subspaces) For $n \in \mathbb{N}$, we define subspace $F_n \subset L(\Lambda_0)$:*

$$F_n := \bigoplus_{j\in\mathbb{Z}} \mathrm{span}\{h(-i_1)\cdots h(-i_k)e^{m\alpha} : 1 \leq \ i_j \leq 2n+j, 0 \leq k \leq n-2j\}. \tag{3.4}$$

Then

$$\chi_{2n-1}(1,1,q) = \mathrm{ch}[F_n](q).$$

In particular, $L(\Lambda_0)$ admits an increasing filtration F_i, for $i \geq 0$,

$$\bigcup_{n\geq 1} F_n = L(\Lambda_0).$$

Proof. From the Frenkel-Kac realization we see with $j \geq 1$

$$e^{j\alpha} = e(-2j+1)\cdots e(-3)e(-1)v_0;\ j \geq 1, \quad e^{-j\alpha} = f(-2j+1)\cdots f(-3)f(-1)v_0.$$

Thus

$$\mathrm{wt}(e^{j\alpha}) = j^2; \quad j \in \mathbb{Z}. \tag{3.5}$$

Also,

$$\mathrm{wt}(h(-i_1)\cdots h(-i_k)v_0) = i_1 + \cdots + i_k.$$

We view q-binomial $\left[\begin{smallmatrix} m+n \\ n \end{smallmatrix}\right]_q$ as a generating function for the partitions in at most n parts whose parts are of size at most m. For any j, the graded dimension of the j-th direct summand in (3.4) equals $\left[\begin{smallmatrix} 2n \\ n+2j \end{smallmatrix}\right]_q$. By summing over all j, together with (3.5), we get

$$\mathrm{ch}[F_n](q) = \sum_{j \in \mathbb{Z}} q^{j^2} \begin{bmatrix} 2n \\ n+2j \end{bmatrix}_q.$$

To finish the proof, we apply (1.5). □

By using the second relation in (1.5) one can prove a similar result for $2\chi_{2n-1}(0,1,q)$ and $L(\Lambda_1)$.

Note that the subspaces F_n are bigraded due to h-grading (by "charge") of $L(\Lambda_0)$. If we use z as the charge variable we immediately get

$$\mathrm{ch}[F_n](z,q) = \sum_{j \in \mathbb{Z}} z^j q^{j^2} \begin{bmatrix} 2n \\ n+2j \end{bmatrix}_q.$$

Example. We have (n=2)

$$\chi_3(1,1,q) = 2q \begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 1 + 3q + 2q^2 + q^3 + q^4.$$

A monomial basis of F_3 is v_0, $e(-1)v_0$, $f(-1)v_0$, $h(-1)v_0$, $h(-1)^2v_0$, $h(-2)v_0$, $h(-2)h(-1)v_0$ and $h(-2)^2v_0$.

We end with several remarks and open questions.

Remark. It would be highly desirable to give a canonical construction of F_n.

It is easy to see (by induction on n) that $\chi_{2n-1}(1,1,1) = \dim(F_n) = 2^{2n-1}$, $n \geq 1$. Interestingly, these are also dimensions of Demazure modules $D(n\theta)$ of $L(\Lambda_0)$. Demazure modules also provide an increasing subspace filtration of $L(\Lambda_0)$, but their structure is different from F_n. At this point we do not see any connection between the two subspaces except for their dimensions.

Remark. There is yet another representation theoretic objects whose character is given by the right hand-side in (3.2) . According to [**5**], the principal subspace $W(\Lambda_0)$ of the vacuum level one $\widehat{\mathfrak{sl}}_3$ basic module satisfies

$$\mathrm{ch}[W(\Lambda_0)] = \sum_{n_1,n_2 \geq 0} \frac{q^{n_1^2+n_2^2-n_1n_2}}{(q;q)_{n_1}(q;q)_{n_2}} = \frac{\sum_{n \in \mathbb{Z}} q^{n^2}}{(q;q)_\infty}. \tag{3.6}$$

This subspace has a combinatorial monomial basis in terms of nilpotent $\widehat{\mathfrak{sl}}_3$ elements. Moreover, in [**4**], q-difference equations and exact sequences are used among certain finite-dimensional subspaces of $W(\Lambda_0)$. It may well be that [**4**] already contains (albeit implicitly) natural candidates for the subspaces $W_n \subset W(\Lambda_0)$ such that, for $n \geq 1$, $\mathrm{ch}[W_n](q) = \chi_{2n-1}(1,1,q)$ and that exact sequences among these subspaces lead to q-recurrences (1.1-1.2) To support this, notice that the second identity in (3.6) was in fact instrumental in Andrews' proof of Wall's conjecture [A]. We leave the problem of identifying W_n for further investigations.

Remark. As a consequence of q-binomial convolution identity we get the following identity

$$\left(-q^{\frac{1}{2}}z;q\right)_n\left(-q^{\frac{1}{2}}z^{-1};q\right)_n = \sum_{j\in\mathbb{Z}} z^n q^{\frac{j^2}{2}} \begin{bmatrix} 2n \\ n+j \end{bmatrix}_q . \tag{3.7}$$

This can be considered as a finite form of (2.1). This identity can be also interpreted as a consequence of "finitized" boson-fermion correspondence in conformal field theory (the left hand-side is "fermionic" while the right hand-side is "bosonic"). Due to similarity of bilateral summations in Andrews' formula for $\chi_{2n-1}(1,1,q)$ and in (3.7) it seems feasible to expect that F_n admits a fermonic realization which would lead to new formulas for the LMW polynomials.

References

[1] G. Andrews, *Partitions, q-series and the Lusztig-Macdonald-Wall conjectures*, Invent. Math. **41** (1977), 91–102.
[2] G. Andrews, *On the Wall polynomials and the L-M-W conjectures*, J. Austral. Math. Soc. Ser. A **37** (1984), 17–26.
[3] G. Andrews, *The theory of partitions*, Cambridge University Press, Cambridge, 1998.
[4] C. Calinescu, J. Lepowsky and A.Milas, *Vertex-algebraic structure of the principal subspaces of level one modules for the untwisted affine Lie algebras of types A,D,E*, Journal of Algebra **323** (2010), 167-192.
[5] B. Feigin and A. Stoyanovsky, *Quasi-particles models for the representations of Lie algebras and geometry of flag manifold*, arXiv preprint (1993) hep-th/9308079.
[6] G. Gasper and M. Rahman, *Basic hypergeometric series, Second edition*, Encycl. of Math. and Applications **96**, Cambridge University Press, Cambridge, 2004.
[7] V. Kac, *Infinite-dimensional Lie algebras*, Cambridge University Press, 1990.
[8] J. Lepowsky and R. Wilson, *Construction of the Affine Lie Algebra* $A_1^{(1)}$, Commun. Math. Phys. **62** (1978), 43-53.
[9] G. Lusztig, *Irreducible representations of finite classical groups*, Invent. Math. **43** (1977), no. 2, 125–175.
[10] G. Wall, *On the conjugacy classes in the unitary, symplectic and orthogonal groups*, J. Austral. Math. Soc. **3** (1963), 1–62.

Mathematical Institute, University of Cologne, Weyertal 86-90, 50931 Cologne, Germany
E-mail address: kbringma@math.uni-koeln.de

Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803, U.S.A.
E-mail address: mahlburg@math.lsu.edu

Department of Mathematics and Statistics, SUNY-Albany, Albany, NY 12222, U.S.A.
E-mail address: amilas@albany.edu