THE OVERPARTITION FUNCTION MODULO SMALL POWERS OF 2

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Abstract. In a recent paper, Fortin, Jacob and Mathieu [5] found congruences modulo powers of 2 for the values of the overpartition function $\overline{p}(n)$ in arithmetic progressions. The moduli for these congruences ranged as high as 64. This note shows that $\overline{p}(n) \equiv 0 \pmod{64}$ for a set of integers of arithmetic density 1.

1. Introduction

An *overpartition* of n is an ordered sequence of nonincreasing integers that sum to n, where the first occurrence of each distinct integer may be overlined. Denote the number of overpartitions of an integer n by $\overline{p}(n)$, with $\overline{p}(0) = 1$ by convention. The generating function for overpartitions is

$$\overline{P}(q) = \sum_{n=0}^{\infty} \overline{p}(n) q^n = \prod_{n=1}^{\infty} \frac{1+q^n}{1-q^n}, \tag{1}$$

which can be understood as a convolution product between partitions into distinct parts (the overlined terms) and ordinary partitions (the rest). Overpartitions have recently been used by Corteel and Lovejoy in combinatorial proofs of many q-series identities, and they arise quite naturally in the study of hypergeometric series (see [3], [4], [10], [12] for details and other references).

They also arise in theoretical physics in the solution of certain problems regarding seas of particles and fields (see [5]). There the authors use an alternative definition of overpartitions that they term *jagged* partitions. A jagged partition of n is an ordered sequence of nonnegative integers $(\lambda_m, \ldots, \lambda_1)$ that sum to n and satisfy the weakly decreasing conditions:

$$\lambda_j \geq \lambda_{j-1} - 1 \qquad \text{and} \qquad \lambda_j \geq \lambda_{j-2}.$$

That this definition is equivalent to overpartitions is a straightforward result.

Proposition 1. *The overpartitions of n correspond bijectively to the jagged partitions of n.*

Proof. This result follows from Euler's Theorem [1], which states that the number of ordinary partitions of an integer m with distinct parts is the same as the number of ordinary partitions of m with only odd parts. We now describe the bijection between jagged partitions and overpartitions. Given a jagged partition $(\lambda_m, \ldots, \lambda_1)$ of n, remove the subset of "increasing" pairs. This is the set of all pairs $(\lambda_j, \lambda_{j-1})$ such that $\lambda_{j-1} = \lambda_j + 1$. Adding the parts in each pair (and maintaining order)

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produces a nonincreasing sequence of odd integers, which becomes a partition into distinct parts by Euler's Theorem. The remaining parts from the original jagged partition form a sequence of nonincreasing integers, i.e., an ordinary partition. Taken together, the distinct part partition and the ordinary partition form an overpartition of n. It is readily apparent that this mapping is bijective. $\square$

In Propositions 11 and 12 of [5], the authors prove several Ramanujan-type congruences modulo small powers of 2 by finding explicit formulas for the generating functions. For example, they prove that for all n,

$$\overline{p}(8n+7) \equiv 0 \pmod{64}.$$

Much more is true, for such congruences are quite common in light of the main result of this note, which describes $\overline{p}(n)$ modulo 64 in terms of representations of integers as sums of squares. In fact, here we prove the following theorem.

Theorem 2. *The set of integers*

$$S = \{n \in \mathbb{Z}_{\geq 0} \mid \quad \overline{p}(n) \equiv 0 \pmod{64}\}$$

has arithmetic density 1.

The proof of the theorem appears shortly in the next section, but first we give an alternative form of the generating function in equation (1). By Jacobi's triple product identity [1], $\overline{P}(q)$ is the inverse of one of Ramanujan's classical theta functions,

$$\theta_4(q) = 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2} = \prod_{n=1}^{\infty}\frac{1-q^n}{1+q^n} = \frac{1}{\overline{P}(q)}. \tag{2}$$

Thus $\overline{P}(q)\cdot\theta_4(q) = 1$. Expanding and iterating this identity as a functional equation for $\overline{P}(q)$ gives the 2-adic expansion found in [5]:

$$\overline{P}(q) = 1 + \sum_{k=1}^{\infty} 2^k \sum_{n_1,\ldots,n_k \geq 1} (-1)^{(n_1+1)\cdots+(n_k+1)} q^{n_1^2+\cdots+n_k^2}. \tag{3}$$

The apparent connection to sums of squares is central in the proof of the theorem.

2. Proof of Theorem 2

Consider the classical theta function

$$\theta(q) = \sum_{n\geq 0} r(n)q^n = \sum_{n=-\infty}^{\infty} q^{n^2}.$$

Then $\theta(q)^k = \sum_{n\geq 0} r_k(n)q^n$ has nonnegative coefficients $r_k(n)$ that count the number of representations of n as the sum of k squares, where different orders and signs are counted distinctly. Note that $r_0(n) = 0$ for any $n \neq 0$, and $r_0(0) = 1$. We will be most interested in a subset of these representations; in particular, let $c_k(n)$ be the number of representations $n = n_1^2 + \cdots + n_k^2$ where each n_i is strictly positive. The two functions are related by the following formula:

$$r_k(n) = 2^k c_k(n) + \sum_{i=1}^{k}\binom{k}{i}(-1)^{i-1} r_{k-i}(n), \tag{4}$$

which is the inclusion-exclusion principle applied to the position of any zeroes in the sums. Classical facts about $r_k(n)$ combined with the combinatorics of $c_k(n)$ will lead to the proof of Theorem 2.

First we record a technical lemma, which is proven by observing that $n^2 \equiv n \pmod 2$ for any integer n.

Lemma 3. *Suppose that two sets of integers $\{n_i \mid 1 \leq i \leq s\}$ and $\{m_j \mid 1 \leq j \leq t\}$ satisfy*

$$n_1^2 + \cdots + n_s^2 = m_1^2 + \cdots + m_t^2.$$

Then $n_1 + \cdots + n_s \equiv m_1 + \cdots + m_t \pmod 2$.

Consider one of the inner sums in equation (3),

$$\sum_{n_1,\ldots,n_k \geq 1} (-1)^{(n_1+1)\cdots+(n_k+1)} q^{n_1^2+\cdots+n_k^2}.$$

For any n, Lemma 3 implies that the sign on every occurrence of q^n is the same. Thus the generating function in (3) may be rewritten as

$$\overline{P}(q) = 1 + \sum_{k=1}^{\infty} 2^k \sum_{n\geq 1} (-1)^{n+k} c_k(n) q^n. \tag{5}$$

Reducing this expression modulo 64 we obtain

$$\begin{aligned}\overline{P}(q) \equiv 1 &+ 2\sum_{n\geq 1}(-1)^{n+1}c_1(n)q^n + 4\sum_{n\geq 1}(-1)^{n}c_2(n)q^n + 8\sum_{n\geq 1}(-1)^{n+1}c_3(n)q^n \\ &+ 16\sum_{n\geq 1}(-1)^{n}c_4(n)q^n + 32\sum_{n\geq 1}(-1)^{n+1}c_5(n)q^n \pmod{64}.\end{aligned} \tag{6}$$

We now argue that in each of the five sums, the coefficient on q^n is zero modulo 64 for a set of arithmetic density 1. This clearly holds for the $c_1(n)$ terms, for they are nonzero only for square integers.

The $c_2(n)$ terms can only be nonzero for those n which are expressible as the sum of two squares, since by definition it is true that $c_k(n) \leq r_k(n)$ for all k and n. Euler showed [9] that an integer n is representable by the sum of two squares if and only if every prime $p \equiv 3 \pmod 4$ occurs with even multiplicity in the factorization of n. Let $B(x)$ denote the number of such integers that are less than or equal to x. It was shown by Landau and later by Ramanujan [8] that

$$B(x) \ll \frac{x}{\sqrt{\log x}}, \tag{7}$$

and thus the integers such that $r_2(n) > 0$ form a set of density 0.

A result of Legendre [9] states that $r_3(n)$ is positive for all integers that are not of the form $4^a(8^b+7)$, which is a set of density 5/6. This is a positive proportion, so we must argue that $c_3(n)$ is almost always divisible by a sufficiently high power of 2. Equation (4) implies that $c_3(n)$ is related to $r_3(n)$ by

$$c_3(n) = \frac{1}{8}(r_3(n) - 3r_2(n) + 3r_1(n) - r_0(n)).$$

As explained above, each of r_2, r_1 and r_0 are nonzero only on a set of density 0, and therefore $c_3(n) = r_3(n)/8$ on a set of density 1.

A theorem of Gauss [7] shows that the values of $r_3(n)$ are intimately related to the Hurwitz class numbers $H(-n)$ of positive definite binary quadratic forms,

$$r_3(n) = \begin{cases} 12H(-4n) & \text{if} \quad n \equiv 1,2,5,6 \pmod 8 \\ 24H(-n) & \text{if} \quad n \equiv 3 \pmod 8 \\ r(n/4) & \text{if} \quad n \equiv 0,4 \pmod 8. \end{cases}$$

Combined with the 8 outside of the sum in (6), we now have that the terms arising from the $c_3(n)$ coefficients are almost always a multiple of 4, and therefore we need an additional multiple of 16 from the class numbers themselves. Gauss also showed that the exponent of 2 that divides $H(-n)$ is given by the number of distinct odd primes dividing the squarefree part of n plus a nonnegative constant that depends on the residue class of n modulo 8. Thus the coefficients in the sum are multiples of 64 for all integers n whose squarefree part contains at least four distinct odd prime factors. Now it must be shown that the set of integers whose squarefree part contains at most three distinct odd prime factors has density zero, but this follows from a classical result in number theory (see [9] for a proof). If $\sigma_k(x)$ denotes the number of integers $n \leq x$ that have at most k distinct odd prime factors, then asymptotically

$$\sigma_k(x) \sim \frac{x(\log\log x)^{k-1}}{(k-1)!\log x}. \tag{8}$$

The density of this set goes to 0 as x grows to infinity, which is the claimed result.

In the fourth sum we must also argue that every coefficient is divisible by a great enough power of 2, for every integer is expressible as the sum of four squares. This is due to Jacobi's result that $r_k(n) > 0$ for all n if $k \geq 4$. In the case of exactly four squares, he also found a simple divisor formula describing the number of such representations [9]. If $\sigma'(n)$ denotes the sum of the divisors of n which are not multiples of 4, then

$$r_4(n) = 8 \sum_{d|n, 4 \nmid d} d = 8\sigma'(n). \tag{9}$$

By equation (4), we can express $c_4(n)$ in terms of the values of $r_k(n)$;

$$c_4(n) = \frac{1}{16}(r_4(n) - 4r_3(n) + 6r_2(n) - 4r_1(n) + r_0(n)). \tag{10}$$

We need to show that $c_4(n)$ is almost always divisible by 4, for there is a factor of 16 outside of the sum in equation (6). As established above, $r_2(n), r_1(n)$ and $r_0(n)$ are nonzero on a set of density 0, and $r_3(n)$ is divisible by 64 on a set of density 1. Thus it remains to show that

$$c_4(n) \equiv \frac{1}{2}\sigma'(n) \equiv 0 \pmod 4$$

for a set of density 1.

Recall that the standard divisor function is multiplicative, which implies that $\sigma'(n)$ is also multiplicative if the even part of n is ignored. Thus if $n = 2^{a_0}p_1^{a_1}\dots p_l^{a_l}$, then

$$\sigma'(n) = C \cdot \sum_{i=0}^{a_1} p_1^i \cdots \sum_{i=0}^{a_l} p_l^i, \tag{11}$$

where $C = 1$ or 3 depending on whether a_0 is zero or positive. Note that if a_i is odd, then the i-th sum in (11) is divisible by 2, and thus $c_4(n)$ is even as long as there are at least three odd primes with odd exponent in the factorization of n. The complement of this set is those integers whose squarefree parts contain at most three distinct odd prime factors, which is a set of density 0 (recall equation (8)).

For the fifth sum, we need only show that $c_5(n)$ is even for a set of density 1. This is accomplished by studying the combinatorics of sums of five positive squares. Let $\vec{s} = (s_1, s_2, s_3, s_4, s_5)$ denote a 5-tuple of positive integers. Then define the set of representations of n as the sum of five squares

$$S_5(n) = \{\vec{s} \in \mathbb{Z}^5_{\geq 1} \mid s_1^2 + s_2^2 + s_3^2 + s_4^2 + s_5^2 = n \text{ and } s_1 \leq s_2 \leq s_3 \leq s_4 \leq s_5\}.$$

The coefficients $c_5(n)$ can now be expressed as

$$c_5(n) = \sum_{\vec{s} \in S_5(n)} m_5(\vec{s}), \tag{12}$$

where $m_5(\vec{s})$ is the number of automorphisms of the set $\{s_1, s_2, s_3, s_4, s_5\}$. The size of this automorphism group is given by a multinomial coefficient that depends on the multiplicities of the parts in $\vec{s}$, in other words,

$$m_5(\vec{s}) = \frac{5!}{\lambda_1! \cdots \lambda_t!}, \tag{13}$$

where $\lambda_t + \cdots + \lambda_1 = 5$ is a partition corresponding to the multiplicities of parts in $\vec{s}$. The expression in (13) is even except for the partition $\{\lambda_2, \lambda_1\} = \{4, 1\}$. Since $c_5(n)$ is a sum of $m_5(\vec{s})$ values, $c_5(n)$ is always even for those n which are not represented by an $\vec{s}$ with multiplicities $\{4, 1\}$. But this partition corresponds to a representation $n = s_1^2 + 4s_2^2 = s_1^2 + (2s_2)^2$. As seen above in (7), the integers n that are representable by two squares form a set of density 0. This completes the proof that $\overline{p}(n) \equiv 0 \pmod{64}$ for a set of density 1.

3. Concluding Remarks

It should be noted that Theorem 2 has a somewhat different flavor than the results found for other common partition functions. A paper of Gordon and Ono [6] proves that if $q(n)$ denotes the number of partitions of n into distinct parts, and k is any positive integer, then $q(n) \equiv 0 \pmod{2^k}$ for a set of integers of arithmetic density 1. Their proof relies on the fact that the generating function for $q(n)$ is a modular function of integral weight, which allows the use of Serre's Theorem regarding the divisibility of the coefficients of such functions [13]. This is in contrast to the situation for the ordinary partition function $p(n)$, where it is still an unproven conjecture (see [11]) that

$$\lim_{X \to \infty} \frac{\#\{n \leq X \quad | \quad p(n) \equiv 0 \pmod{2}\}}{X} = \frac{1}{2}.$$

The difference is that the generating function for $p(n)$ is a modular form of half-integral weight. The coefficients for such functions are poorly understood, and conjectures such as this are considered to be very difficult with present techniques.

In the current setting, we have that $\overline{P}(q)$ is a modular form of half-integral weight, and thus we were unable to use general theorems about the coefficients. Fortunately, we still obtained stronger results than those for the ordinary partition function by investigating the combinatorial properties of the generating function and its connection to well-known theta functions, whose nonzero coefficients are sparse and simply described. In fact, preliminary calculations suggest that Theorem 2 may also hold for arbitrary powers of 2. However, the proof of a more general result will likely require a different approach, for the methods used in this paper run into serious limitations beyond the modulus of 64.

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