PARTITION CONGRUENCES AND THE ANDREWS-GARVAN-DYSON CRANK

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ABSTRACT. In 1944, Freeman Dyson conjectured the existence of a "crank" function for partitions that would provide a combinatorial proof of Ramanujan's congruence modulo 11. Forty years later, Andrews and Garvan successfully found such a function, and proved the celebrated result that the crank simultaneously "explains" the three Ramanujan congruences modulo $5, 7$ and 11. This note announces the proof of a conjecture of Ono, which essentially asserts that the elusive crank satisfies exactly the same types of general congruences as the partition function.

1. Introduction and statement of results

Whether these guesses are warranted by the evidence, I leave to the reader to decide. Whatever the final verdict of posterity may be, I believe the 'crank' is unique among arithmetical functions in having been named before it was discovered. May it be preserved from the ignominious fate of the planet Vulcan! (Freeman Dyson [1])

A *partition* of n is a non-increasing list of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ that sum to n; we write $|\lambda| = n$. The partition function $p(n)$ is defined to count the number of distinct partitions of a given integer n.

Ramanujan's celebrated congruences for the partition function state that

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5}, \\ p(7n+5) &\equiv 0 \pmod{7}, \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned} \tag{1.1}$$

Following the spirit of Ramanujan's own work, Watson and Atkin extended these congruences to arbitrary powers of $5, 7$ and 11 [2]. Sporadic progress was made in proving congruences for primes up to 31, until Ono's seminal paper from 2000 achieved a surprising improvement [3]. He proved the existence of infinite families of partition congruences for every prime $\ell \geq 5$ by developing the p-adic theory of half-integral weight modular forms. This result was expanded to include congruences for every

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modulus coprime to 6 by Ahlgren and Ono [4, 5]. These results are typically much more complicated than Ramanujan's original congruences, as displayed by the example

$$p(48037937n + 1122838) \equiv 0 \pmod{17}. \tag{1.2}$$

However, there is another side to the story, which began when Freeman Dyson wondered whether a simple statistic might group the partitions into natural classes and explain the Ramanujan congruences. The *rank* of the partition $\lambda_1 + \lambda_2 + \cdots + \lambda_k$, is defined by

$$\mathrm{rank}(\lambda) := \lambda_1 - k. \tag{1.3}$$

Dyson observed empirically that this function decomposes the Ramanujan congruences modulo 5 and 7 into classes of equal size [1]. For example,

$$\mathcal{N}(m, 5, 5n+4) = \frac{1}{5} \cdot p(5n+4) \qquad 0 \leq m \leq 4, \tag{1.4}$$

where $\mathcal{N}(m, N, n)$ is the number of partitions λ of n for which $\mathrm{rank}(\lambda) \equiv m \pmod{N}$. His observations were proven 10 years later by Atkin and Swinnerton-Dyer [6]. However, even the smallest examples show that the rank does not equally dissect the Ramanujan congruence modulo 11.

Instead, Dyson conjectured that there would be a "crank" function for the final Ramanujan congruence, although it wasn't until forty years had passed that Andrews and Garvan defined the function and showed that

$$\mathcal{M}(m, 11, 11n+6) = \frac{1}{11} \cdot p(11n+6). \tag{1.5}$$

Here $\mathcal{M}(m, N, n)$ is defined for the crank just as $\mathcal{N}(m, N, n)$ was for the rank [7, 8]. In these historic works, they also showed that the crank dissects the Ramanujan congruences modulo 5 and 7 in a different way than the rank.

If $\lambda_1 + \lambda_2 + \cdots + \lambda_s + 1 + \cdots + 1$ has exactly r ones, then let $o(\lambda)$ be the number of parts of λ that are strictly larger than r. The *crank* is given by

$$\mathrm{crank}(\lambda) := \begin{cases} \lambda_1 & \text{if} \quad r = 0, \\ o(\lambda) - r & \text{if} \quad r \geq 1. \end{cases} \tag{1.6}$$

Clearly $p(n) = \mathcal{M}(0, N, n) + \cdots + \mathcal{M}(N-1, N, n)$, and for the Ramanujan congruences, all of these summands are equal. However, this behavior is atypical, and an unpublished conjecture of Ono asserted that a different approach would show that congruences for the partition function are related to the crank in a universal manner.

Conjecture (Ono). *For every prime $\ell \geq 5$ and integer $\tau \geq 1$, there are infinitely many non-nested arithmetic progressions $An + B$ for which*

$$\mathcal{M}(m, \ell, An+B) = 0 \pmod{\ell^\tau}$$

for every $0 \leq m \leq \ell - 1$.

In fact, the following theorem shows that the crank function actually satisfies congruences beyond those predicted by Ono.

Theorem 1.1. *Suppose that $\ell \geq 5$ is prime and that τ and j are positive integers. Then there are infinitely many non-nested arithmetic progressions $An+B$ such that*

$$\mathcal{M}(m, \ell^j, An+B) \equiv 0 \pmod{\ell^\tau}$$

simultaneously for every $0 \leq m \leq \ell^j - 1$.

Remark. The frequency of such congruences is quantified later in this note by Theorem 4.1. An obvious implication of Theorem 1.1 is that $p(An+B) \equiv 0 \pmod{\ell^\tau}$ as well, corresponding to the congruences found by Ahlgren and Ono.

This announcement begins with a review of modular forms in Section 2. Following that, Section 3 explains how to write the generating function of the crank in terms of Klein forms. A condensed proof of Theorem 1.1 is found in Section 4.

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2. Half-integral weight modular forms

This section contains the basic definitions and properties of modular forms that will be needed in Section 4 (see [2] for details). Let $\Gamma := SL_2(\mathbb{Z})$ denote the full modular group of 2-by-2 matrices, and for a given modulus N, let $\Gamma_0(N)$ and $\Gamma_1(N)$ denote the subsets of matrices that are congruent to $\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) \pmod N$, respectively. If $k \in \frac{1}{2}\mathbb{Z}$, and $\Gamma' \subset \Gamma$ is such a congruence subgroup, then $M_k^!(\Gamma')$ denotes the vector space of nearly holomorphic modular forms of weight k for the subgroup Γ' (these are holomorphic on the upper half-plane $\mathcal{H}$, and meromorphic at the cusps of Γ'). The forms that are holomorphic at the cusps are denoted by $M_k(\Gamma')$, and the forms that vanish at the cusps by $S_k(\Gamma')$. If $\Gamma' = \Gamma_0(N)$, then $M_k^!(\Gamma_0(N), \chi)$, $M_k(\Gamma_0(N), \chi)$, or $S_k(\Gamma_0(N), \chi)$ denotes the appropriate space of modular forms of weight k on $\Gamma_0(N)$ with Nebentypus character χ.

For a meromorphic function f on the upper half plane $\mathcal{H}$, and an integer k, the *slash operator* is defined by $f(z) \mid_k \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) := (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right)$, for any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. A key property is that this is a group action in the sense that for any $M_1, M_2 \in \Gamma$, $(f(z) \mid_k M_1) \mid_k M_2 = f(z) \mid_k M_1 M_2$.

Let $q := e^{2\pi i z}$. If $f = \sum_{n=0}^{\infty} a(n) q^n \in M_k(\Gamma_0(N), \chi)$ and ψ is a Dirichlet character, then the *twist* of f by ψ is

$$f(z) \otimes \psi := \sum_{n \geq 0} \psi(n) a(n) q^n. \tag{2.1}$$

Simple facts about Gauss sums allow one to rewrite the twist of a modular form using the slash operator in a manner that is independent of the weight. If p is a prime and $g_p := \sum_{v=1}^{p-1} (\frac{v}{p}) e^{2\pi i v/p}$ is the standard Gauss sum, then

$$f(z) \otimes \left(\frac{\bullet}{p}\right) = \frac{g_p}{p} \sum_{v=1}^{p-1} \left(\frac{v}{p}\right) f(z) \,\Big|\, \begin{pmatrix} 1 & -v/p \\ 0 & 1 \end{pmatrix}. \tag{2.2}$$

The half-integral weight Hecke operators are important tools for finding congruences among the coefficients of modular forms. If $f(z) = \sum_{n \geq 0} a(n) q^n \in M_k(\Gamma_0(N), \chi)$ and k is not an integer, then for a prime $p \nmid N$ the Hecke operator is defined by

$$f(z) \mid T(p^2) := \sum_{n \geq 0} \left(a(p^2 n) + \chi^*(p) \left(\frac{n}{p}\right) p^{k-3/2} a(n) + \chi^*(p^2) p^{2k-2} a\left(\frac{n}{p^2}\right) \right) q^n, \tag{2.3}$$

where $\chi^*(n) := \chi(n) \left(\frac{(-1)^{k-1/2}}{n}\right)$.

All of these operators act on spaces of modular forms in an easily described manner.

Proposition 2.1. *Suppose that* $f(z) = \sum_{n \geq 0} a(n) q^n \in M_k(\Gamma_0(N), \chi)$.

1. *For a prime* $p \nmid N$ *and a non-integral half-integer* k*, the action of* $T(p^2)$ *is space-preserving, i.e.,*
$$f(z) \mid T(p^2) \in M_k(\Gamma_0(N), \chi).$$
2. *If* ψ *is a character with modulus* M*, then*
$$f(z) \otimes \psi \in M_k(\Gamma_0(NM^2), \chi\psi^2).$$
3. *If* $f(z)$ *is a cusp form,* $t \geq 1$ *and* $0 \leq r \leq t-1$*, then*
$$\sum_{n \equiv r \pmod{t}} a(n) q^n \in S_k(\Gamma_1(Nt^2)).$$

In Section 4 we will need to simultaneously find congruences for two half-integral weight modular forms of different weights and levels. This is partially addressed by Ono's Theorem 2.2 in [9], Ahlgren and Ono's proof of Lemma 3.1 in [5], and Serre's arguments in [10]. The additional ingredients needed to prove the next theorem are the integral weight Hecke operators, the famous Shimura correspondence, and the decomposition $S_k(\Gamma_1(N)) = \oplus S_k(\Gamma_0(N), \chi)$, where the sum is over all even characters χ.

Theorem 2.2. *Suppose that* k_i *and* N_i *are positive integers for* $1 \leq i \leq r$*, and let* $g_1(z), \ldots, g_r(z)$ *be half-integer weight cusp forms with algebraic integer coefficients such that* $g_i(z) \in S_{k_i+1/2}(\Gamma_1(N_i))$*. If* $M \geq 1$*, then a positive proportion of primes* $p \equiv -1 \pmod{N_1 \cdots N_r M}$ *have the property that for every* i*,*

$$g_i(z) \mid T(p^2) \equiv 0 \pmod{M}.$$

3. The crank generating function and Klein forms

Let $\mathcal{M}(m,n)$ be the number of partitions λ of n such that $\text{crank}(\lambda)=m$. Using the generating function found by Andrews and Garvan [8], define

$$F(x,z) := \sum_{m=-\infty}^{\infty}\sum_{n\geq 0}\mathcal{M}(m,n)x^m q^n = \sum_{\lambda} x^{\text{crank}(\lambda)} q^{|\lambda|} \tag{3.1}$$
$$= \prod_{n\geq 1}\frac{1-q^n}{(1-xq^n)(1-x^{-1}q^n)}.$$

Consider a positive integer N and set $\zeta := e^{2\pi i/N}$. For any residue class $m \pmod N$, elementary calculations give the generating function for the crank,

$$\sum_{n\geq 0}\mathcal{M}(m,N,n)q^n = \frac{1}{N}\sum_{s=0}^{N-1}F(\zeta^s,z)\zeta^{-ms} \tag{3.2}$$
$$= \frac{1}{N}\sum_{s=0}^{N-1}\zeta^{-ms}\left(\prod_{n\geq 1}\frac{(1-q^n)}{(1-\zeta^{-s}q^n)(1-\zeta^s q^n)}\right).$$

To prove congruences for this function, we need to show that it is a modular form. Recall Dedekind's eta-function

$$\eta(z) := q^{1/24}\prod_{n\geq 1}(1-q^n), \tag{3.3}$$

which is a modular form of weight 1/2. Perhaps less familiar are the Klein forms, which were studied extensively by Kubert and Lang [11].

Definition 3.1. Let $1\leq s\leq N-1$. The $(0,s)$–*Klein form* is given by

$$t_{0,s}(z) := \frac{\omega_s}{2\pi i}\cdot\prod_{n\geq 1}\frac{(1-\zeta^s q^n)(1-\zeta^{-s}q^n)}{(1-q^n)^2},$$

where $\omega_s := \zeta^{s/2}(1-\zeta^{-s})$.

Now write $\overline{d}$ for the least residue of d modulo N and set $\exp(z) := e^{2\pi i z}$. Understanding the action of Γ on the Klein forms is an important aspect of the proof of Theorem 1.1. The following formula comes from equation **K2** on page 28 of [11].

Proposition 3.2. *If* $\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \Gamma_0(N)$, *then*

$$t_{0,s}(z)\Big|_{-1}\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \beta\cdot t_{0,\overline{ds}}(z),$$

where β *is a root of unity given by* $\beta := \exp\left(\frac{cs+(ds-\overline{ds})}{2N} - \frac{cds^2}{2N^2}\right)$.

A simple calculation shows that this "multiplier system" is always trivial for a certain congruence subgroup.

Corollary 3.3. *If* $1 \leq s \leq N-1$*, then* $t_{0,s}(z) \in M^!_{-1}(\Gamma_1(2N^2))$.

Returning to the crank generating function, for $1 \leq s \leq N-1$ equation (3.1) becomes

$$F(\zeta^s, z) = \frac{1}{\eta(z) t_{0,s}(z)} \cdot \frac{\omega_s q^{1/24}}{2\pi i}. \tag{3.4}$$

The generating function for the partition function is $\sum_{n\geq 0} p(n)q^n = \prod_{n\geq 1} \frac{1}{(1-q^n)}$, which is the $s = 0$ term in (3.2). Hence

$$\sum_{n\geq 0} \mathcal{M}(m, N, n)q^n = \frac{1}{2\pi i N} \sum_{s=1}^{N-1} \frac{\omega_s \zeta^{-ms}}{\eta(z) t_{0,s}(z)} \cdot q^{1/24} + \frac{1}{N} \sum_{n\geq 0} p(n)q^n. \tag{3.5}$$

4. The proof of Theorem 1.1

For a prime $\ell \geq 5$, set $\delta_\ell := (\ell^2 - 1)/24$, and define $\epsilon_\ell := \left(\frac{\delta_\ell}{\ell}\right)$. Then define the set

$$S_\ell := \left\{ 0 \leq \beta \leq \ell - 1 \;\Big|\; \left(\frac{\beta + \delta_\ell}{\ell}\right) = 0 \text{ or } -\epsilon_\ell \right\}. \tag{4.1}$$

The following theorem is a more precise description of the congruences satisfied by the crank function, and it clearly implies Theorem 1.1.

Theorem 4.1. *Suppose that* $\ell \geq 5$ *is prime,* τ *and* j *are positive integers, and* $\beta \in S_\ell$*. Then a positive proportion of primes* $Q \equiv -1 \pmod{24\ell}$ *have the property that for every* $0 \leq m \leq \ell^j - 1$,

$$\mathcal{M}\left(m, \ell^j, \frac{Q^3 n + 1}{24}\right) \equiv 0 \pmod{\ell^\tau}$$

for all $n \equiv 1 - 24\beta \pmod{24\ell}$ *that are not divisible by* Q.

For the rest of this section, let $N := \ell^j$ be a fixed power of a fixed prime $\ell \geq 5$. Theorem 2.2 applies to modular forms with algebraic integer coefficients, and thus (3.5) must be rescaled in defining

$$\begin{aligned} g_m(z) &:= \left(\sum_{n\geq 0} N \cdot \mathcal{M}(m, N, n) q^{n+\delta_\ell}\right) \prod_{n\geq 1} (1 - q^{\ell n})^\ell \\ &= \frac{1}{2\pi i} \sum_{s=1}^{N-1} \frac{\eta^\ell(\ell z)}{\eta(z)} \cdot \frac{\omega_s \zeta^{-ms}}{t_{0,s}(z)} + \frac{n^\ell(\ell z)}{\eta(z)}. \end{aligned} \tag{4.2}$$

Let $G_m(z)$ and $P(z)$, respectively, denote the two summands in the final expression of (4.2).

As explained in [5], if t is a positive integer, then there is a Dirichlet character $\chi_{\ell,t}$ such that

$$E_t(z) := \frac{\eta^{\ell^t}(z)}{\eta(\ell^t z)} \in M_{\frac{\ell^t-1}{2}}(\Gamma_0(\ell^t), \chi_{\ell,t}). \tag{4.3}$$

This form vanishes at every cusp $\frac{a}{c}$ with ℓ^t not dividing c, and also satisfies $E_t(z)^{\ell^\tau} \equiv 1 \pmod{\ell^{\tau+1}}$ for any $\tau \geq 0$. Similar facts about eta-quotients, along with Corollary 3.3, show that

$$P(z) = \frac{\eta^{\ell}(\ell z)}{\eta(z)} \in M_{\frac{\ell-1}{2}}\left(\Gamma_0(\ell), \left(\frac{\bullet}{\ell}\right)\right), \quad \text{and} \tag{4.4}$$

$$G_m(z) \in M^!_{\frac{\ell+1}{2}}(\Gamma_1(2N^2)). \tag{4.5}$$

For any function $f(z)$ with a q-expansion, define $\widetilde{f}(z) := f(z) - \epsilon_\ell \cdot f(z) \otimes \left(\frac{\bullet}{\ell}\right)$. The line of argument in [5] implies that if τ is sufficiently large, then there is some integer $\lambda' \geq 1$ and some character χ such that

$$\frac{\widetilde{P}(24z)}{\eta^{\ell}(24\ell z)} \cdot E_{j+1}(24z)^{\ell^\tau} \in S_{\lambda'+1/2}(\Gamma_0(576\ell^{\max\{3,j+1\}}), \chi). \tag{4.6}$$

We now show that a similar cusp form exists for $G_m(z)$.

Lemma 4.2. *If τ is sufficiently large, then there is some $\lambda \geq 1$ such that*

$$\frac{\widetilde{G_m}(24z)}{\eta^{\ell}(24\ell z)} \cdot E_{j+1}(24z)^{\ell^\tau} \in S_{\lambda+1/2}(\Gamma_1(576\ell^2 N^2)). \tag{4.7}$$

Proof. Recalling equation (4.3), if τ sufficiently large, then it only needs to be shown that $\widetilde{G_m}(z)/\eta^{\ell}(\ell z)$ vanishes at each cusp $\frac{a}{c}$ with $\ell N \mid c$. If $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(\ell N)$, then the expansion of $1/\eta^{\ell}(\ell z)$ at that cusp is $(q^{-\ell^2/24} + \dots)$ up to a root of unity. Thus it must be proven that the expansion of $\widetilde{G_m}(z)$ at $\frac{a}{c}$ is $(*q^h + \dots)$ for some $h > \ell^2/24$.

Using Proposition 3.2 and equation (4.4), calculate

$$G_m(z) \mid_{\frac{\ell+1}{2}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{2\pi i} \left(\frac{d}{\ell}\right) \frac{\eta^{\ell}(\ell z)}{\eta(z)} \sum_{s=1}^{N-1} \frac{\omega_s \zeta^{-ms}}{\beta_s t_{0,\overline{ds}}(z)}, \tag{4.8}$$

where the β_s are the roots of unity described by Proposition 3.2.

To find the expansion of $G_m(z) \otimes \left(\frac{\bullet}{\ell}\right)$, first observe that for any v'

$$\begin{pmatrix} 1 & -v/\ell \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} 1 & -v'/\ell \\ 0 & 1 \end{pmatrix}, \qquad \text{where} \tag{4.9}$$

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} := \begin{pmatrix} a - cv/\ell & b - cvv'/\ell^2 + (av' - dv)/\ell \\ c & d + cv'/\ell \end{pmatrix}.$$

Pick $v' \equiv d^2 v \pmod{\ell}$ for the subsequent arguments, and let g_ℓ be the Gauss sum as in (2.2). Proposition 3.2 and equations (4.8) and (4.9) show that

$$\left(G_m(z) \otimes \left(\frac{\bullet}{\ell}\right)\right) \Big|_{\frac{\ell+1}{2}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{g_\ell}{2\pi i \ell} \left(\frac{d'}{\ell}\right) \sum_{s=1}^{N-1} \sum_{v=1}^{\ell-1} \left(\frac{v}{\ell}\right) \frac{\eta^\ell(\ell z)}{\eta(z)} \cdot \frac{\omega_s \zeta^{-ms}}{\beta'_s t_{0,\overline{d's}}(z)} \Big| \begin{pmatrix} 1 & -v'/\ell \\ 0 & 1 \end{pmatrix}. \tag{4.10}$$

Since $d' \equiv d \pmod{N}$ and $\ell N \mid c$, a short computation verifies that $\beta'_s = \beta_s$. An additional calculation then shows that the first term of (4.8) is ϵ_ℓ times the first term of (4.10). Thus the expansion of $\widetilde{G_m}(z)$ has the form $(*q^{\delta_\ell+1} + \dots)$, and $\delta_\ell + 1 > \ell^2/24$. □

The following theorem is now proved, as $F'(z)$ is defined by (4.6), and $F_m(z)$ is described by Lemma 4.2.

Theorem 4.3. *For any $\tau \geq 0$ and $0 \leq m \leq N-1$, there is a character χ, positive integers λ and λ', and modular forms $F_m(z) \in S_{\lambda+1/2}(\Gamma_1(576\ell^2N^2))$ and $F'(z) \in S_{\lambda'+1/2}(\Gamma_0(576\ell^{\max\{3,j+1\}}), \chi)$ such that*

$$\frac{\widetilde{g_m}(24z)}{\eta^\ell(24\ell z)} \equiv F_m(z) + F'(z) \pmod{\ell^\tau}.$$

Deduction of Theorem 4.1 from Theorem 4.3. Suppose that $\beta \in S_\ell$. Starting from the definition of $g_m(z)$ in (4.2), restrict $\widetilde{g_m}(24z)/\eta^\ell(24\ell z)$ to those indices $n' \equiv \beta + \delta_\ell \pmod{\ell}$, which then gives a new series (scaled by a factor of 1/2 when $\beta \not\equiv -\delta_\ell \pmod{\ell}$)

$$h_{m,\beta}(z) := \sum_{n' \equiv \beta+\delta_\ell \pmod{\ell}} N\mathcal{M}(m, N, n' - \delta_\ell) q^{24n'-\ell^2} = \sum_{n \equiv 24\beta-1 \pmod{24\ell}} N\mathcal{M}\left(m, N, \frac{n+1}{24}\right) q^n. \tag{4.11}$$

But Theorem 4.3 implies that

$$h_{m,\beta}(z) \equiv F_{m,\beta}(z) + F'_\beta(z) \pmod{\ell^\tau}, \tag{4.12}$$

where $F_{m,\beta}(z)$ and $F'_\beta(z)$ are defined by restricting $F_m(z)$ and $F'(z)$ to only those indices with $n' \equiv \beta + \delta_\ell \pmod{\ell}$. Proposition 2.1 and Theorem 2.2 then show that a positive proportion of primes $Q \equiv -1 \pmod{24\ell}$ have the property that

$$F_{m,\beta}(z) \mid T(Q^2) \equiv F'_\beta(z) \mid T(Q^2) \equiv 0 \pmod{\ell^\tau}$$

for all m. Replace n by Qn in (2.3) to see that

$$N\mathcal{M}\left(m, N, \frac{Q^3 n + 1}{24}\right) \equiv 0 \pmod{\ell^\tau} \tag{4.13}$$

for all $n \equiv 1-24\beta \pmod{24\ell}$ that are not divisible by Q. Dividing by N then completes the proof. □

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