18.781 Exam 2 Practice Problems - Fall 2008

These problems are a sample of the material that may appear on Exam 2, and are meant as extra practice.

1. If $p \geq 3$ is prime, how many solutions are there to $x^a \equiv 1 \pmod{p}$ as a function of a?

2. Determine how many solutions each of the following equations has, and find them if there are any.

   (a) $x^{11} \equiv 22 \pmod{23}$

   (b) $x^6 \equiv 9 \pmod{17}$

   (c) $x^{13} \equiv 13 \pmod{29}$

   (d) $x^{21} \equiv 15 \pmod{29}$.

3. Find a primitive root modulo 13^2.

4. Use the binomial theorem to show that $\sum_{k=0}^{n} \binom{n}{k} 2^n = 3^n$.

5. How many solutions are there to $2x^3 - x^2 - 1 \equiv 0 \pmod{125}$?

6. How many solutions are there to $(x^2 - 3)(y^2 - 2) \equiv 0 \pmod{p}$ as a function of p?

7. If $n = p_1^{a_1} \cdots p_r^{a_r}$, how many quadratic residues modulo n are there?

8. Find all of the quadratic residues modulo 31 and 35.

9. (a) Determine whether $2x^2 - 3x + 7 \equiv 0 \pmod{131}$ is solvable.

   (b) Determine whether $x^2 \equiv 46 \pmod{91}$ is solvable (note that 91 is not prime).

10. Characterize all primes p such that $\left(\frac{18}{p}\right) = 1$.

11. Evaluate the Legendre/Jacobi symbols:

    (a) $\left(\frac{31}{103}\right)$

    (b) $\left(\frac{21}{73}\right)$

    (c) $\left(\frac{-15}{69}\right)$

    (d) $\left(\frac{18}{100}\right)$

    (e) $\left(\frac{200}{97}\right)$.

12. For any odd integer n, evaluate $\left(\frac{(n-1)(n+1)}{n}\right)$.

13. Use quadratic reciprocity to characterize the primes for which $x^2 - 3y^2 \equiv 0 \pmod{p}$ is definitely not solvable.

14. Find infinitely many solutions to $x^2 - 7y^2 = 1$.

15. Expand the following fractions into simple continued fractions:

(a) $\dfrac{25}{7}$

(b) $\dfrac{7}{25}$

(c) $\dfrac{48}{17}$.

16. Prove that if $x = [a_0, a_1, \ldots, a_r]$ is greater than 1, then $\frac{1}{x} = [0, a_0, a_1, \ldots, a_r]$.

17. Convert the continued fractions into rational numbers:

    (a) $[-2, 5, 1, 3]$

    (b) $[4, 3, 2, 1]$

    (c) $[1, 2, 3, 4]$.

18. Calculate $[3, 6, 3, 1, 2]$ by recursively finding all of its convergents.

19. (a) Evaluate $[2, 3, 2, 3, 2, 3, \ldots]$

    (b) Evaluate $[3, 2, 3, 2, 3, 2, \ldots]$.