18.781 Final Exam Practice Problems - Fall 2008
(excluding material from Exam 2)

These problems are a representative sample of the questions that may appear on the Final Exam, and are meant as extra practice. Note that this is a cumulative exam, and will include material from throughout the semester.

1. (a) Calculate the greatest common divisor of 588 and 1356.
   (b) Find all integers x and y such that $588x + 1356y = 36$.
2. Calculate the least common multiples of the following sets of integers:
   (a) 13 and 24,
   (b) 108 and 222,
   (c) 34 and 52,
   (d) $n, n+1$, and $n+2$, where $n \equiv 1 \pmod 4$.
3. (a) Suppose that $(a, n) = 1$ and $d \mid n$. Is it then true $(a + kd, n) = 1$ for any k? If not, can you characterize the k for which it is true?
   (b) Given that $972 = 6 \cdot 162$, evaluate $(163, 972)$.
4. One euro is currently worth approximately \$1.30. Suppose that a traveler returns with his pockets full of euro coins only to discover that the exchange office only has \$20 bills. What is the minimum number of euros that can be exchanged exactly for bills (with no wasted change leftover)?
5. Use the binomial theorem to prove that if p is prime and $a^p \equiv a \pmod p$, then $(a+1)^p \equiv a + 1 \pmod p$. (*This actually leads to an inductive proof of Fermat's Little Theorem!*)
6. (a) Prove that if a prime $p \mid (n^2 - n + 1)$ for some integer n, then $p \equiv 1 \pmod 6$.
   (b) More generally, prove that if a prime $p = a^2 + ab + b^2$, then $p \equiv 1 \pmod 6$.
7. Suppose that $n \equiv 7 \pmod 8$.
   (a) Prove that there is a prime divisor $p \mid n$ such that $p \not\equiv 1 \pmod 8$.
   (b) Must there necessarily a prime divisor such that $p \equiv 7 \pmod 8$?
8. Prove that $33 \mid (n^{10} - 1)$ whenever n is coprime to 33.
9. Find all solutions of the congruences
   (a) $18x \equiv 8 \pmod{34}$,
   (b) $24x \equiv 15 \pmod{96}$,
   (c) $17x \equiv 7 \pmod{21}$.
10. Find all solutions to the congruences $x \equiv 2 \pmod 5$, $x \equiv 1 \pmod 6$, $x \equiv 37 \pmod{55}$.
11. Find the smallest positive integer n that is a multiple of 10 but is not divisible by $3, 7$, or 13.

12. Suppose that f is a multiplicative function satisfying $f(4) = 8$, $f(5) = 5$, and $f(28) = 56$. Calculate $f(35)$.

13. Evaluate the sum $\sum_{d|405} \mu(d)d^2$.

14. How many integers from 1 to 92 are coprime to 92?

15. Use the theory of multiplicative functions to answer the following questions.

    (a) How many distinct positive divisors does 3960 have?

    (b) What is the sum of all of its divisors?

16. (a) Verify that the Carmichael number 561 is a pseudoprime for the base 3 (use efficient successive squaring).

    (b) Is 561 a strong probable prime for the base 3?

17. Calculate the first several terms in the continued fraction expansions of

    (a) $\sqrt[3]{5}$, (b) 2π.

18. If $\theta_1 = [2, 1, 4, 11, a_1, a_2, \ldots]$ and $\theta_2 = [2, 1, 4, 8, b_1, b_2, \ldots]$, find an upper bound for the difference $|\theta_1 - \theta_2|$.

19. Find the solutions of $x^2 - 6y^2 = \pm 1$ by first calculating the continued fraction expansion of $\sqrt{6}$.

20. (a) Are $\frac{4}{7}$ and $\frac{13}{22}$ adjacent in any Farey sequence? If not, find a fraction with a smaller denominator that lies in between them.

    (b) Given that $\frac{2}{5}$ and $\frac{3}{7}$ are adjacent in $\mathcal{F}_7$, find the portion of the Farey sequence $\mathcal{F}_{22}$ that lies in the range $[\frac{2}{5}, \frac{3}{7}]$.