

18.781 Problem Set 1 - Fall 2008

Due Tuesday, Sep. 16 at 1:00

1. (Niven 5.3.3) Find all PT's whose terms form an

- (a) Arithmetic progression
- (b) Geometric progression

2. (Niven 5.3.7) For which n are there solutions to $x^2 - y^2 = n$?

3. (Niven 5.3.9) Prove that any integer n can be expressed in the form

$$n = x^2 + y^2 - z^2$$

(contrast this with Gauss' 3-squares Theorem!).

4. Find all PPT's with $c = a + 2$.

5. The n -th *triangular number* is given by $T_n := 1 + 2 + \dots + n$. The first few values are $T_1 = 1, T_2 = 3, T_3 = 6, \dots$

- (a) Prove using induction that $T_n = \frac{n(n+1)}{2}$.
- (b) Prove that for any n there is a PPT containing $4T_n$. For example, $(7, 24, 25)$ contains $4T_3 = 24$.

6. Prove that

$$\sum_{i=1}^n \frac{1}{i(i+1)} = \frac{n}{n+1}.$$

7. (a) Find all rational points on the circle $x^2 + y^2 = 2$, using $(1, 1)$ as the starting point.

(b) Try to use the same method to find all rational points on $x^2 + y^2 = 3$. What goes wrong?

(Bonus) Prove that there are no rational points on $x^2 + y^2 = 3$.

8. (Niven 1.2.2) Find the greatest common divisor $g = (1819, 3587)$, and find x, y such that

$$1819x + 3587y = g.$$

9. (Niven 1.2.9) Show that if $ac \mid bc$, then $a \mid b$.

10. (Niven 1.2.10) Show that if $a \mid b$ and $c \mid d$, then $ac \mid bd$.

11. (Niven 1.2.11) Prove that $4 \nmid (n^2 + 2)$ for any n .

12. (Niven 1.2.12) Given that $(a, 4) = (b, 4) = 2$, prove that $(a + b, 4) = 4$.

13. (Niven 1.2.17) Evaluate $(n, n + 1)$ and $[n, n + 1]$.

14. (Niven 1.2.36) Prove that $(a, b, c) = ((a, b), c)$.

15. (Niven 1.2.43) Prove that $a \mid bc$ if and only if $\frac{a}{(a,b)} \mid c$.

16. Prove that in the Euclidean algorithm, $r_{i+2} < \frac{1}{2}r_i$.

(Bonus) Find a bound on the total number of steps in the algorithm.

17. (Niven 1.2.45) Prove that any positive integer a can be uniquely expressed as

$$a = 3^m + b_{m-1}3^{m-1} + \cdots + b_13 + b_0,$$

where $b_i = 0, 1,$ or -1 .