18.781 Problem Set 10 - Fall 2008

Due Tuesday, Nov. 25 at 1:00

1. (Niven 7.4.4) (*Tails don't matter.*) Consider the following phenomenon for decimal approximations: if we pick a string of arbitrary digits, e.g. 466832... and append them to the decimal truncations of $\sqrt{2}$, then the sequence

$$\begin{aligned} &\mathbf{1}.466832\ldots \\ &\mathbf{1.4}466832\ldots \\ &\mathbf{1.41}466832\ldots \\ &\mathbf{1.414}466832\ldots \end{aligned}$$

converges to $\sqrt{2}$ regardless of the appended digits.

Show that the same holds for continued fractions: if $\theta = [a_0, a_1, \ldots]$, and $b_1, b_2, \ldots$ is any sequence of positive integers, prove that

$$\lim_{n\to\infty} [a_0, a_1, \ldots, a_n, b_1, b_2, \ldots] = \theta.$$

2. (Niven 7.5.6) Suppose that $\theta = [a_0, a_1, \ldots]$ is an irrational simple continued fraction. In this problem you will describe the continued fraction expansion of $-\theta$.

   (a) Show that $-\theta = [-a_0, -a_1, -a_2, \ldots]$.
   Hint: Write $\theta_n = a_n + \frac{1}{\theta_{n+1}}$, *with* $\theta_n := [a_n, a_{n+1}, \ldots]$, *and use induction.*

   (b) Show that if $a_1 > 1$,

   $$-\theta = [-a_0 - 1, 1, a_1 - 1, a_2, a_3, \ldots],$$

   and if $a_1 = 1$,

   $$-\theta = [-a_0 - 1, a_2 + 1, a_3, \ldots].$$

   Hint: Expand $-\theta = -[a_0, a_1, \theta_2]$ *and compare to the expressions above.*

   (c) An important theorem (Thm. 7.10 in Niven) states that each irrational number is uniquely expressible as a simple continued fraction. Explain why parts (a) and (b) do not contradict this fact.

3. Prove that $\frac{13}{9}$ is a convergent of $\sqrt[3]{3}$ by checking that the approximation is sufficiently close.

4. (*Periodic convergents.*) In this problem you will explore a different set of convergents of infinite continued fractions. Suppose that $\xi = [a_0, a_1, \ldots]$, and define the periodic convergents by $\xi_n := [\overline{a_0, a_1, \ldots, a_n}]$.

   (a) If $h_n = a_n h_{n-1} + h_{n-2}$ and $k_n = a_n k_{n-1} + k_{n-2}$ as usual, show that the periodic convergents satisfy the quadratic equations

   $$k_n \xi_n^2 - (h_n - k_{n-1})\xi - h_{n-1} = 0.$$

(b) Recall the standard finite convergents $r_n = \frac{h_n}{k_n} = [a_0, \ldots, a_n]$ and prove that

$$|\xi_n - r_n| < \frac{1}{k_n k_{n-1}}.$$

Use the convergence of the r_n to conclude that $\lim_{n\to\infty} \xi_n = \xi$ as well.

(c) Part (b) implies that the ξ_n form a sequence of quadratic irrationals that are better and better approximations of ξ. Calculate the first three periodic convergents of $\pi = [3, 7, 15, 1, 292, \ldots]$.

5. (Niven 7.7.3) Expand $\sqrt{15}$ into an infinite simple continued fraction (try to do it without a calculator first!).

6. Use a calculator to expand $\frac{13+3\sqrt{11}}{7}$ into an infinite simple continued fraction. Once you have obtained an answer, check that it is correct by solving the the resulting quadratic equation.

7. (Niven 7.8.8) Given that $\sqrt{18} = [4, \overline{4, 8}]$, find the least positive solution of $x^2 - 18y^2 = -1$ (if any), and of $x^2 - 18y^2 = 1$.

8. Is the number $3.82842712474619\ldots$ likely to be a quadratic irrational? If so, identify which one, and check that it matches all given digits.
(*Hint: Calculate the first several terms in the continued fraction expansion.*)