18.781 Solutions to Problem Set 10 - Fall 2008

Due Tuesday, Nov. 25 at 1:00

1. (Niven 7.4.4) (*Tails don't matter.*) Consider the following phenomenon for decimal approximations: if we pick a string of arbitrary digits, e.g. $466832\ldots$ and append them to the decimal truncations of $\sqrt{2}$, then the sequence

$$\mathbf{1}.466832\ldots$$
$$\mathbf{1.4}466832\ldots$$
$$\mathbf{1.41}466832\ldots$$
$$\mathbf{1.414}466832\ldots$$

converges to $\sqrt{2}$ regardless of the appended digits.

Show that the same holds for continued fractions: if $\theta = [a_0, a_1, \ldots]$, and $b_1, b_2, \ldots$ is any sequence of positive integers, prove that

$$\lim_{n\to\infty} [a_0, a_1, \ldots, a_n, b_1, b_2, \ldots] = \theta.$$

Let θ_n be the infinite continued fraction $[a_0, a_1, \ldots, a_n, b_1, b_2, \ldots]$. We have that the difference of θ_n and $r_n = \frac{h_n}{k_n} = [a_0, a_1, \ldots, a_n]$ is given by the formula involving the tail $\beta = [b_1, b_2, \ldots]$ of the series:

$$\theta_n = \frac{\beta h_n + h_{n-1}}{\beta k_n + k_{n-1}}.$$

So the difference of θ_n and r_n is

$$|\theta_n - r_n| = \left| \frac{\beta h_n + h_{n-1}}{\beta k_n + k_{n-1}} - \frac{h_n}{k_n} \right| = \left| \frac{\pm 1}{(\beta k_n + k_{n-1})k_n} \right|.$$

Therefore the difference between θ_n and r_n has limit zero as $n \longrightarrow \infty$. Since, by definition, θ is the limit of r_n, this implies it is also the limit of θ_n.

2. (Niven 7.5.6) Suppose that $\theta = [a_0, a_1, \ldots]$ is an irrational simple continued fraction. In this problem you will describe the continued fraction expansion of $-\theta$.

   (a) Show that $-\theta = [-a_0, -a_1, -a_2, \ldots]$.
   Hint: Write $\theta_n = a_n + \frac{1}{\theta_{n+1}}$, with $\theta_n := [a_n, a_{n+1}, \ldots]$, and use induction.
   I proved this for finite continued fractions, using induction on the length. For the base case, it is easy to see that $-[a_0] = [-a_0]$. Now if we assume for any length k continued fraction that $-[a_0, a_1, \ldots, a_{k-1}] = [-a_0, -a_1, \ldots, -a_{k-1}]$. Now we take a length $k+1$ fraction, $[b_0, b_1, \ldots, b_n]$. We have

$$-[b_0, \ldots, b_k] = -b_0 + \frac{1}{-[b_1, \ldots, b_k]} = -b_0 + \frac{1}{[-b_1, \ldots, -b_k]} = [-b_0, \ldots, -b_k].$$

   This shows the result for finite continued fractions. Now for the continued fraction of θ, we just use the limit definition:

$$-\theta = \lim_{n\longrightarrow\infty} -[a_0, a_1, \ldots, a_n] = \lim_{n\longrightarrow\infty}[-a_0, -a_1, \ldots, -a_n] = [-a_0, -a_1, \ldots].$$

(b) Show that if $a_1 > 1$,

$$-\theta = [-a_0 - 1, 1, a_1 - 1, a_2, a_3, \ldots],$$

and if $a_1 = 1$,

$$-\theta = [-a_0 - 1, a_2 + 1, a_3, \ldots].$$

Hint: Expand $-\theta = -[a_0, a_1, \theta_2]$ and compare to the expressions above.

Say $a_1 > 1$. Then

$$[-a_0 - 1, 1, a_1 - 1, a_2, a_3, \ldots] = [-a_0 - 1, 1, (\theta - a_0)^{-1} - 1] = [-a_0 - 1, 1 + \frac{\theta - a_0}{a_0 + 1 - \theta}]$$
$$= -a_0 - 1 + (a_0 + 1 - \theta) = -\theta.$$

If instead $a_1 = 1$, then

$$[-a_0 - 1, a_2 + 1, a_3 \ldots] = [-a_0 - 1, ((\theta - a_0)^{-1} - 1)^{-1} + 1] = -a_0 - 1 + \cfrac{1}{1 + \cfrac{1}{-1 + \cfrac{1}{\theta - a_0}}}$$

$$= -a_0 - 1 + \cfrac{1}{1 + \cfrac{\theta - a_0}{a_0 + 1 - \theta}} = -\theta.$$

(c) An important theorem (Thm. 7.10 in Niven) states that each irrational number is uniquely expressible as a simple continued fraction. Explain why parts (a) and (b) do not contradict this fact.

The theorem assumes the continued fraction are simple, that is, they have the form $[a_0, a_1, \ldots, a_n, \ldots]$ where a_i is a positive integer for all $i > 0$, and a_0 is any integer. Part (a) has negative integers in all entries, so is not simple. Part (b) is in fact the unique way to write $-\theta$ as a simple continued fraction.

3. Prove that $\frac{13}{9}$ is a convergent of $\sqrt[3]{3}$ by checking that the approximation is sufficiently close.

We can check that the approximation is sufficiently close *without* using a calculator by the following method:

$$\left(\frac{13}{9} - \sqrt[3]{3}\right)\left(\frac{13^2}{9^2} + \frac{13\sqrt[3]{3}}{9} + \sqrt[3]{9}\right) = \frac{13^3}{9^3} - 3 = \frac{10}{729}.$$

Now, we can bound the second factor below by $3(\frac{4}{3})^2 = \frac{16}{3}$, since $\frac{4}{3}$ is clearly larger than the cube root and the fraction. This in turn gives the following bounds on the first factor:

$$0 < \frac{13}{9} - \sqrt[3]{3} < \frac{5}{1944} < \frac{1}{2 \cdot 9^2}.$$

So the fraction must appear in the continued fraction expansion of $\sqrt[3]{3}$.

4. (*Periodic convergents.*) In this problem you will explore a different set of convergents of infinite continued fractions. Suppose that $\xi = [a_0, a_1, \ldots]$, and define the periodic convergents by $\xi_n := [\overline{a_0, a_1, \ldots, a_n}]$.

   (a) If $h_n = a_n h_{n-1} + h_{n-2}$ and $k_n = a_n k_{n-1} + k_{n-2}$ as usual, show that the periodic convergents satisfy the quadratic equations

   $$k_n \xi_n^2 - (h_n - k_{n-1})\xi - h_{n-1} = 0.$$

   We use the same formula involving the tail of the continued fraction we used above, but here we exploit the repeating nature of the fraction entries. In particular, we have that $\xi_n = [a_0, a_1, \ldots, a_n, \xi_n]$, so the formula gives

   $$\xi_n = \frac{\xi_n h_n + h_1}{\xi_n k_n + k_{n-1}}.$$

   Clearing the denominator and collecting terms gives the quadratic equation above.

   (b) Recall the standard finite convergents $r_n = \frac{h_n}{k_n} = [a_0, \ldots, a_n]$ and prove that

   $$|\xi_n - r_n| < \frac{1}{k_n k_{n-1}}.$$

   Use the convergence of the r_n to conclude that $\lim_{n\to\infty} \xi_n = \xi$ as well.

   So we use the same formula as above:

   $$|\xi_n - r_n| = \left| \frac{\xi_n h_n + h_1}{\xi_n k_n + k_{n-1}} - \frac{h_n}{k_n} \right| = \frac{1}{(\xi_n k_n + k_{n-1})k_n} < \frac{1}{k_{n-1}k_n}.$$

   Because this difference goes to zero as n increases to infinity, the two sequences have the same limit, so we get that

   $$\lim_{n\longrightarrow\infty} \xi_n = \lim_{n\longrightarrow\infty} r_n = \xi$$

   (c) Part (b) implies that the ξ_n form a sequence of quadratic irrationals that are better and better approximations of ξ. Calculate the first three periodic convergents of $\pi = [3, 7, 15, 1, 292, \ldots]$.

   The periodic convergents of π are $[\overline{3}]$, $[\overline{3, 7}]$, and $[\overline{3, 7, 15}]$. These satisfy the respective quadratic equations $x^2 - 3x - 1 = 0, 7x^2 - 21x - 3 = 0$, and $106x^2 - 325x - 25 = 0$, and so are equal to

   $$\frac{3 + \sqrt{13}}{2}, \frac{21 + 5\sqrt{21}}{14}, \text{ and } \frac{325 + 5\sqrt{4649}}{212}.$$

5. (Niven 7.7.3) Expand $\sqrt{15}$ into an infinite simple continued fraction (try to do it without a calculator first!).

   Without a calculator, you can do the following calculation:

   $$\sqrt{15} = 3 + (\sqrt{15} - 3)$$
   $$\frac{1}{\sqrt{15} - 3} = \frac{\sqrt{15} + 3}{6} = 1 + \frac{\sqrt{15} - 3}{6}$$
   $$\frac{6}{\sqrt{15} - 3} = \frac{6(\sqrt{15} + 3)}{6} = 6 + (\sqrt{15} - 3)$$

   This repeated remainder gives the expansion $\sqrt{15} = [3, \overline{1, 6}]$.

6. Use a calculator to expand $\frac{13+3\sqrt{11}}{7}$ into an infinite simple continued fraction. Once you have obtained an answer, check that it is correct by solving the the resulting quadratic equation.

   If you calculated the continued fraction correctly, you should've gotten that

$$\frac{13+3\sqrt{11}}{7} = [3, \overline{3, 1, 1, 2}].$$

   So, letting $y = [\overline{3, 1, 1, 2}]$, we get the equation

$$y = 3 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{y}}}}.$$

   This comes to

$$y = \frac{18y+7}{5y+2} \Longrightarrow 5y^2 - 16y - 7 = 0 \Longrightarrow y = \frac{8+3\sqrt{11}}{5}$$

   .

   Now, we let $x = [3, \overline{3, 1, 1, 2}] = 3 + \frac{1}{y} = \frac{13+3\sqrt{11}}{7}$.

7. (Niven 7.8.8) Given that $\sqrt{18} = [4, \overline{4, 8}]$, find the least positive solution of $x^2 - 18y^2 = -1$ (if any), and of $x^2 - 18y^2 = 1$.

   Looking at the first equation mod 3, we can see that it has no integer solutions, since $\left(\frac{-1}{3}\right) = -1$.

   Now, using convergents to $\sqrt{18}$, we get the first two convergents are 4 and $\frac{17}{4}$. We check $4^2 - 18(1)^2 = -2$, but $17^2 - 18(4)^2 = 1$. So this is the smallest integer solution of Pell's equation for $d = 18$.

8. Is the number $3.82842712474619\ldots$ likely to be a quadratic irrational? If so, identify which one, and check that it matches all given digits.
   (*Hint: Calculate the first several terms in the continued fraction expansion.*)

   The first complete continued fraction for the decimal digits shown is

$$[3, 1, 4, 1, 4, 1, 4, 1, 4, 1, 4, 1, 4, 1, 4, 1, 4, 1, 4, 1, 4, 1, 1, 1, 2, 1, 8, 1, 2, 2, 2, 3, 1, 1, 3, 4, 1, 8].$$

   This looks like the infinite decimal might represent $[3, \overline{1, 4}]$. Calculating the quadratic irrational with this continued fraction, we get $1 + 2\sqrt{2}$. This does match the decimal expansion given above.