18.781 Problem Set 11 - Fall 2008
Due Thursday, Dec. 4 at 1:00

Throughout this assignment, $\mathcal{F}_n$ denotes the Farey sequence of order n.

1. Write the complete Farey sequence of order 7, $\mathcal{F}_7$.

2. (Niven 6.1.1) Let $\frac{a}{b}$ and $\frac{a'}{b'}$ be the left and right neighbors (respectively) of $\frac{1}{2}$ in $\mathcal{F}_n$. Prove that b is the greatest odd integer less than n, and that $a + a' = b$.

3. (Niven 6.1.2) Let $S_n := 1 + \sum_{k=1}^{n} \phi(k)$.
   (a) Prove that $\mathcal{F}_n$ consists of S_n distinct fractions.
   (b) Prove that the sum of all of the fractions in $\mathcal{F}_n$ is $S_n/2$.

4. (Niven 6.1.4) Suppose that $\frac{a}{b}$ and $\frac{a'}{b'}$ are any two adjacent fractions in $\mathcal{F}_n$.
   (a) Prove that $\left|\frac{a}{b} - \frac{a'}{b'}\right| \geq \frac{1}{n(n-1)}$.
   (b) Prove that $\left|\frac{a}{b} - \frac{a'}{b'}\right| \leq \frac{1}{n}$.
   (c) Prove that both bounds are actually achieved by some choice of fractions.

5. (Niven 6.1.7 & 6.1.8)
   (a) Let $b_1, b_2, \ldots, b_s$ be the denominators of all fractions in $\mathcal{F}_n$ read from left to right. Prove that
   $$\sum_{k=1}^{s-1} \frac{1}{b_k b_{k+1}} = 1.$$
   Hint: Place the Farey sequence on the unit interval $[0, 1]$ *and consider the distance between each successive fraction.*
   (b) Show that
   $$\sum_{b,b'} \frac{1}{bb'} = 1,$$
   where the sum is taken over all $1 \leq b, b' \leq n$ that satisfy $(b, b') = 1$ and $b + b' > n$. *Hint: Consider the mediants of* $\mathcal{F}_n$.

6. *Continued fractions.* If $\frac{a}{b} < \frac{k}{n} < \frac{a'}{b'}$ in $\mathcal{F}_n$, we define the *neighbors* of $\frac{k}{n}$ as the two surrounding fractions $\frac{a}{b}$ and $\frac{a'}{b'}$, and the *children* of $\frac{k}{n}$ as the mediants $\frac{a+k}{b+n}$ and $\frac{a'+k}{b'+n}$.

   For example, the fractions $\frac{1}{3}\ \frac{2}{5}\ \frac{1}{2}$ are part of the sequence $\mathcal{F}_5$. Therefore the neighbors of $\frac{2}{5}$ are $\frac{1}{3}$ and $\frac{1}{2}$, and its children are $\frac{3}{8}$ and $\frac{3}{7}$.
   (a) Show that the neighbors of $\frac{a+k}{b+n}$ are $\frac{a}{b}$ and $\frac{k}{n}$, and that its children are $\frac{2a+k}{2b+n}$ and $\frac{a+2k}{b+2n}$.

(b) Prove that the simple continued fractions $[a_0, a_1, \ldots, a_{r-1}]$ and $[a_0, a_1, \ldots, a_{r-1}, a_r]$ are adjacent in some Farey sequence.

(c) As a simple consequence of (b), prove that $[a_0, a_1, \ldots, a_r]$ and $[a_0, a_1, \ldots, a_r + 1]$ are adjacent in some Farey sequence.

(Bonus) Prove inductively that if $\frac{k}{n}$ has the continued fraction expansion $[a_0, a_1, \ldots, a_r]$ with $a_r > 1$, then its neighbors are $[a_0, a_1, \ldots, a_{r-1}]$ and $[a_0, a_1, \ldots, a_r - 1]$. Then prove that its children are $[a_0, a_1, \ldots, a_r + 1]$ and $[a_0, a_1, \ldots, a_r - 1, 2]$.

7. (Niven 6.1.9) The *Ford circles* of order n (denoted $\mathcal{C}_n$) are the circles of radius $\frac{1}{2b^2}$ that are tangent to the x-axis at the fraction $\frac{a}{b} \in \mathcal{F}_n$. Prove that if $\frac{a}{b}$ and $\frac{a'}{b'}$ are adjacent Farey fractions, then the corresponding Ford circles are tangent.

8. (Niven 6.2.6) Suppose that an irrational number x lies between two consecutive fractions $\frac{a}{b}$ and $\frac{a'}{b'}$ in $\mathcal{F}_n$. Prove that either

$$\left| x - \frac{a}{b} \right| < \frac{1}{2b^2} \qquad \text{or} \qquad \left| x - \frac{a}{b'} \right| < \frac{1}{2b'^2}.$$

9. *Direct construction of $\mathcal{F}_n$.* Suppose that $\frac{a}{b}$ is in $\mathcal{F}_n$. This problem describes a simple algorithm for finding the next fraction in the sequence (the algorithm actually requires keeping track of the two previous fractions).

(a) Recall that $\frac{a'}{b'}$ is adjacent (on the right) to $\frac{a}{b}$ in some Farey sequence if and only if $a'b - ab' = 1$. Show that if $b + b' \leq n$, then $\frac{a}{b}$ and $\frac{a'}{b'}$ are not adjacent in $\mathcal{F}_n$.

(b) Show that $\frac{a'+ka}{b'+kb}$ is adjacent to $\frac{a}{b}$ in $\mathcal{F}_n$ if and only if $b' + kb \leq n < b' + (k+1)b$.

(c) Explain how the Euclidean algorithm and part (b) can be used to find $\frac{a'}{b'}$ adjacent to $\frac{a}{b}$.

(d) *Algorithm:* Using part (c), we now have $\frac{a}{b}$ and $\frac{a'}{b'}$ adjacent (in order) in $\mathcal{F}_n$. Let $k := \lfloor \frac{n+b}{b'} \rfloor$. Prove that $\frac{c}{d} := \frac{ka'-a}{kb'-b}$ is then adjacent to $\frac{a'}{b'}$.
Remark: The Euclidean algorithm is unnecessary in all subsequent steps, as we directly compute the next fraction using the previous two!

(e) Beginning from $\frac{3}{8}$, calculate the next five terms in $\mathcal{F}_{13}$.

10. Prove that $\alpha = \sqrt{7} + \sqrt{5}$ is an algebraic number by finding a polynomial $f(x)$ with integral coefficients such that $f(\alpha) = 0$.