18.781 Solutions to Problem Set 11 - Fall 2008

Due Thursday, Dec. 4 at 1:00

Throughout this assignment, $\mathcal{F}_n$ denotes the Farey sequence of order n.

1. Write the complete Farey sequence of order 7, $\mathcal{F}_7$.

   Here's a straightforward problem! We get:

$$\frac{0}{1}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{1}{1}.$$

2. (Niven 6.1.1) Let $\frac{a}{b}$ and $\frac{a'}{b'}$ be the left and right neighbors (respectively) of $\frac{1}{2}$ in $\mathcal{F}_n$. Prove that b is the greatest odd integer less than n, and that $a + a' = b$.

   From our knowledge of mediants and adjacent fractions in $\mathcal{F}_n$, we know that for $\frac{a}{b}$ to be the left neighbor of $\frac{1}{2}$, we must have that $b - 2a = 1$ and $b + 2 > n \geq b$. The equation shows that b is odd, and the inequality shows that it is the greatest odd integer less than or equal to n. The respective relations for $\frac{a'}{b'}$ are $2a' - b' = 1$ and $b' + 2 > n \geq b'$. These imply that $b = b'$, and then subtracting the equalities we get $2b - 2a - 2a' = 0$, which gives $a + a' = b$.

3. (Niven 6.1.2) Let $S_n := 1 + \sum_{k=1}^{n} \phi(k)$.

   (a) Prove that $\mathcal{F}_n$ consists of S_n distinct fractions.

   This is done by induction. First we note that $\mathcal{F}_1$ has two fractions, $\frac{0}{1}$ and $\frac{1}{1}$, and $S_n = 1 + \phi(1) = 2$. So the base case is satisfied.

   Now assume that S_{k-1} is the number of fractions in $\mathcal{F}_{k-1}$. Then $\mathcal{F}_k$ consists of all the fractions in $\mathcal{F}_{k-1}$ and all the fractions between zero and one that have k as their denominator when written in lowest terms. The possible numerators are then all integers between 1 and k that are relatively prime to k, which is $\phi(k)$. Thus, from our inductive hypothesis,

$$\#(\mathcal{F}_k) = \#(\mathcal{F}_{k-1}) + \phi(k) = (1 + \sum_{i=1}^{k-1} \phi(i)) + \phi(k) = S_k.$$

   So our result is proven, by induction.

   (b) Prove that the sum of all of the fractions in $\mathcal{F}_n$ is $S_n/2$.

   Consider all the fractions in $\mathcal{F}_n$ that are not $\frac{1}{2}$. These can be grouped into pairs that sum to 1, because $\frac{a}{b}$ is in $\mathcal{F}_n$ exactly when $\frac{b-a}{b}$ is, since $(a, b) = (b - a, b)$. Since these pairs sum to 1, the sum of all fractions not equal to $\frac{1}{2}$ is equal to this number of pairs, which is half the number of fractions not equal to $\frac{1}{2}$. Now, the only fraction that could've been left out is equal to $\frac{1}{2}$, so the sum of all the terms in $\mathcal{F}_n$ is half the number of terms, or $\frac{S_n}{2}$.

4. (Niven 6.1.4) Suppose that $\frac{a}{b}$ and $\frac{a'}{b'}$ are any two adjacent fractions in $\mathcal{F}_n$.

(a) Prove that $\left|\frac{a}{b} - \frac{a'}{b'}\right| \geq \frac{1}{n(n-1)}$. (*Assumption*: $n > 1$.)

We have that $|ab' - a'b| = 1$ and so $(b, b') = 1$, so

$$\left|\frac{a}{b} - \frac{a'}{b'}\right| = \frac{|ab' - a'b|}{|bb'|} = \frac{1}{bb'}.$$

But we know b and b' are both less than or equal to n. Since they are also relatively prime, the largest their product can be is $n(n-1)$. Thus the smallest this fraction could be is $\frac{1}{n(n-1)}$.

(b) Prove that $\left|\frac{a}{b} - \frac{a'}{b'}\right| \leq \frac{1}{n}$.

Using the same equality above, now we need to bound bb' from below. We know b and b' are two positive integers that sum to a number greater than n (otherwise their mediant would be in $\mathcal{F}_n$. Minimizing xy for $x + y \geq n + 1$, $x, y \in \mathbb{Z}^+$ is easily shown to be $\{x, y\} = \{1, n\}$, so the product is n, which corresponds to the bound given.

(c) Prove that both bounds are actually achieved by some choice of fractions.

Sometimes success is in the first place you look. In this case, start with the first three fractions in $\mathcal{F}_n$, $\frac{0}{1}, \frac{1}{n}, \frac{1}{n-1}$. The first two have the largest possible difference, and the second two have the smallest.

5. (Niven 6.1.7 & 6.1.8)

(a) Let $b_1, b_2, \ldots, b_s$ be the denominators of all fractions in $\mathcal{F}_n$ read from left to right. Prove that

$$\sum_{k=1}^{s-1} \frac{1}{b_k b_{k+1}} = 1.$$

Hint: Place the Farey sequence on the unit interval $[0, 1]$ *and consider the distance between each successive fraction.*

Above, we showed that the difference between two successive fractions $\frac{a}{b}$ and $\frac{a'}{b'}$ is exactly $\frac{1}{bb'}$. Thus it only depends on the denominators. Let's sequence the numerators of the fractions in $\mathcal{F}_n$ as $a_1, a_2, \ldots, a_s$. Then we have that

$$\sum_{k=1}^{s-1} \frac{1}{b_k b_{k+1}} = \sum_{k=1}^{s-1} \left(\frac{a_{k+1}}{b_{k+1}} - \frac{a_k}{b_k}\right) = \frac{a_s}{b_s} - \frac{a_1}{b_1} = \frac{1}{1} - \frac{0}{1} = 1.$$

(b) Show that

$$\sum_{b,b'} \frac{1}{bb'} = 1,$$

where the sum is taken over all $1 \leq b, b' \leq n$ that satisfy $(b, b') = 1$ and $b + b' > n$. *Hint: Consider the mediants of* $\mathcal{F}_n$.

The trick here is to show that this sum is the same as the previous one. This involves showing that the indices match up in a one-to-one correspondence. The hint says to consider mediants, indeed, the gaps between consectutive terms can be indexed by their mediants. We know that two consecutive terms, $\frac{a}{b}$ and $\frac{a'}{b'}$, have the mediant $\frac{a+a'}{b+b'}$, and $b+b'>n$, since this fraction is not in $\mathcal{F}_n$. Also, the two denominators are relatively prime, since we know $ab'-a'b=\pm 1$. So given two consecutive fractions in the sum in the previous problem, $\frac{a}{b}<\frac{a'}{b'}$, we can map that to the choice b, b' in the indices in this sum.

Now we have to show the other direction, that any choice of b, b' in this sum associates to a unique pair of consecutive fractions in $\mathcal{F}_n$. So we need to find a, a'. This comes from knowing that we can solve the equation $a'b-ab'=1$. The assumptions about b, b' is that $(b, b')=1$, $b+b'>n$, and $1\le b, b'\le n$. So the first implies that this equation is solvable by some a_0 and a_0'. Now all the solutions of $a'b-ab'=1$ are given by a_0+bt and $a_0'+b't$, for $t\in\mathbb{Z}$. So we can choose a so that $0\le a<b$. This implies

$$\frac{1}{b}\le a'<b'+\frac{1}{b}, \text{ or } 0<a'\le b'.$$

So the fractions $\frac{a}{b}$ and $\frac{a'}{b'}$ have mediant $\frac{a+a'}{b+b'}$. Since $b+b'>n$, they are consecutive in $\mathcal{F}_n$, so we have shown that the indices match up.

6. *Continued fractions.* If $\frac{a}{b}<\frac{k}{n}<\frac{a'}{b'}$ in $\mathcal{F}_n$, we define the *neighbors* of $\frac{k}{n}$ as the two surrounding fractions $\frac{a}{b}$ and $\frac{a'}{b'}$, and the *children* of $\frac{k}{n}$ as the mediants $\frac{a+k}{b+n}$ and $\frac{a'+k}{b'+n}$.

For example, the fractions $\frac{1}{3}\ \frac{2}{5}\ \frac{1}{2}$ are part of the sequence $\mathcal{F}_5$. Therefore the neighbors of $\frac{2}{5}$ are $\frac{1}{3}$ and $\frac{1}{2}$, and its children are $\frac{3}{8}$ and $\frac{3}{7}$.

(a) Show that the neighbors of $\frac{a+k}{b+n}$ are $\frac{a}{b}$ and $\frac{k}{n}$, and that its children are $\frac{2a+k}{2b+n}$ and $\frac{a+2k}{b+2n}$.

We know that the mediant of $\frac{a}{b}$ and $\frac{k}{n}$ is $\frac{a+k}{b+n}$, so in $\mathcal{F}_{b+n}$, the sequence $\frac{a}{b}, \frac{a+k}{b+n}, \frac{k}{n}$ occurs. This shows the neighbors of $\frac{a+k}{b+n}$ are as stated. Also, the children of $\frac{a+k}{b+n}$ are simply the mediants to either side of the fraction, so can be calculated by summing numerators and denominators with either neighbor. We get $\frac{2a+k}{2b+n}$ to the left and $a+2kb+2n$ on the right.

(b) Prove that the simple continued fractions $[a_0, a_1, \dots, a_{r-1}]$ and $[a_0, a_1, \dots, a_{r-1}, a_r]$ are adjacent in some Farey sequence.

These are successive convergents, so we can write them as $\dfrac{h_{r-1}}{k_{r-1}}$ and $\dfrac{h_r}{k_r}$. From our knowledge of continued fractions, we know these satisfy

$$|h_{r-1}k_r-h_rk_{r-1}|=1.$$

Furthermore, in $\mathcal{F}_{k_r}$, we have that the denominators are both less than or equal to k_r and the sum is greater than k_r, so by the arguments in the previous problem, the fractions are consecutive in $\mathcal{F}_{k_r}$.

(c) As a simple consequence of (b), prove that $[a_0, a_1, \dots, a_r]$ and $[a_0, a_1, \dots, a_r+1]$ are adjacent in some Farey sequence.

The fraction $[a_0, a_1, \dots, a_r+1]$ equals the fraction $[a_0, a_1, \dots, a_r, 1]$, since

$$a_r+1=a_r+\frac{1}{1}.$$

Writing the second fraction this way, we can apply the previous problem to and conclude that the fractions are consecutive in some Farey sequence.

(Bonus) Prove inductively that if $\frac{k}{n}$ has the continued fraction expansion $[a_0, a_1, \ldots, a_r]$ with $a_r > 1$, then its neighbors are $[a_0, a_1, \ldots, a_{r-1}]$ and $[a_0, a_1, \ldots, a_r - 1]$. Then prove that its children are $[a_0, a_1, \ldots, a_r + 1]$ and $[a_0, a_1, \ldots, a_r - 1, 2]$.

7. (Niven 6.1.9) The *Ford circles* of order n (denoted $\mathcal{C}_n$) are the circles of radius $\frac{1}{2b^2}$ that are tangent to the x-axis at the fraction $\frac{a}{b} \in \mathcal{F}_n$. Prove that if $\frac{a}{b}$ and $\frac{a'}{b'}$ are adjacent Farey fractions, then the corresponding Ford circles are tangent.

   The important part of this problem is to draw the picture correctly: the two circles that are drawn should be resting on top of the number line, touching at the two fractions. Then we can use the Pythagorean theorem to find the distance between the centers of the two circles, which are at the points $(\frac{a}{b}, \frac{1}{2b^2})$ and $(\frac{a'}{b'} . \frac{1}{2b'^2})$. We get that this distance is

$$\left[\left(\frac{a}{b} - \frac{a'}{b'}\right)^2 + \left(\frac{1}{2b^2} - \frac{1}{2b'^2}\right)^2\right]^{\frac{1}{2}} = \left(\frac{1}{b^2b'^2} + \left(\frac{1}{4b^4} - \frac{1}{2b^2b'^2} + \frac{1}{4b'^4}\right)\right)^{\frac{1}{2}}$$
$$= \left(\frac{1}{4b^4} + \frac{1}{2b^2b'^2} + \frac{1}{4b'^4}\right)^{\frac{1}{2}} = \frac{1}{2b^2} + \frac{1}{2b'^2}.$$

   So the distance between the two centers of circles is equal to the sum of their radii, so the circles are tangent.

8. (Niven 6.2.6) Suppose that an irrational number x lies between two consecutive fractions $\frac{a}{b}$ and $\frac{a'}{b'}$ in $\mathcal{F}_n$. Prove that either

$$\left|x - \frac{a}{b}\right| < \frac{1}{2b^2} \qquad \text{or} \qquad \left|x - \frac{a'}{b'}\right| < \frac{1}{2b'^2}.$$

   We can use the previous problem. Since the two Ford circles that lie above the neighboring fractions of x are tangent, at least one of them lies above x. This implies that x is within the distance of the radius from the corresponding fraction on the number line. But that is exactly what the above inequalities state.

9. *Direct construction of* $\mathcal{F}_n$. Suppose that $\frac{a}{b}$ is in $\mathcal{F}_n$. This problem describes a simple algorithm for finding the next fraction in the sequence (the algorithm actually requires keeping track of the two previous fractions).

   (a) Recall that $\frac{a'}{b'}$ is adjacent (on the right) to $\frac{a}{b}$ in some Farey sequence if and only if $a'b - ab' = 1$. Show that if $b + b' \le n$, then $\frac{a}{b}$ and $\frac{a'}{b'}$ are not adjacent in $\mathcal{F}_n$.

   This comes from the fact that if $b+b' \le n$, then the mediant of the fractions, $\frac{a+a'}{b+b'}$, has been inserted between them in $\mathcal{F}_n$.

   (b) Show that $\frac{a'+ka}{b'+kb}$ is adjacent to $\frac{a}{b}$ in $\mathcal{F}_n$ if and only if $b' + kb \le n < b' + (k+1)b$.

   We have that $\frac{a'+ka}{b'+kb}$ is adjacent to $\frac{a}{b}$ in some Farey sequence since $(a' + ka)b - (b' + kb)a = 1$. Also, their mediant has denominator $b' + (k+1)b$. So they are adjacent in exactly the Farey sequences $\mathcal{F}_n$ with $b' + kb \le n < b' + (k+1)b$.

(c) Explain how the Euclidean algorithm and part (b) can be used to find $\frac{a'}{b'}$ adjacent to $\frac{a}{b}$.
So we can use the Euclidean algoritm to solve the equation $bx - ay = 1$ for x and y in $\mathbb{Z}$. Now, for the fractions to be adjacent in $\mathcal{F}_n$, we want y to be such that $n - b < y \leq n$, which can be guaranteed by adding multiples of b to it while adding multiples of a to x. Then if we have that solution, we let $a' = x$ and $b' = y$. Then from the above work, $\frac{a'}{b'}$ is the next term in $\mathcal{F}_n$.

(d) *Algorithm:* Using part (c), we now have $\frac{a}{b}$ and $\frac{a'}{b'}$ adjacent (in order) in $\mathcal{F}_n$. Let $k := \lfloor \frac{n+b}{b'} \rfloor$. Prove that $\frac{c}{d} := \frac{ka'-a}{kb'-b}$ is then adjacent to $\frac{a'}{b'}$.
Remark: The Euclidean algorithm is unnecessary in all subsequent steps, as we directly compute the next fraction using the previous two!
So here we just have to show that it satisfies the equations and inequalities above. We have that

$$(ka' - a)b' - (kb' - b)a' = a'b - b'a = 1$$

and since $n + b - b' < kb' \leq n + b$, we have

$$n - b' < kb' - b \leq n.$$

So from the above work, we are done.

(e) Beginning from $\frac{3}{8}$, calculate the next five terms in $\mathcal{F}_{13}$.
From the Euclidean algorithm we get that the next fraction in $\mathcal{F}_8$ is $\frac{2}{5}$. But we note that the mediant of these is in $\mathcal{F}_{13}$, so the next fraction in $\mathcal{F}_{13}$ is $\frac{5}{13}$. Then we use part (d) to find the sequence:

$$\frac{3}{8}, \frac{5}{13}, \frac{2}{5}, \frac{5}{12}, \frac{3}{7}, \frac{4}{9}, \ldots$$

10. Prove that $\alpha = \sqrt{7} + \sqrt{5}$ is an algebraic number by finding a polynomial $f(x)$ with integral coefficients such that $f(\alpha) = 0$.

The easiest way to do this is to find a linear combination of the powers of α that equal zero. We have that the powers of α are

$$1, \sqrt{5} + \sqrt{7}, 12 + 2\sqrt{35}, 26\sqrt{5} + 22\sqrt{7}, 284 + 48\sqrt{35}, \ldots$$

We note that $1, 12 + 2\sqrt{35}$, and $284 + 48\sqrt{35}$ can be combined to get 0:

$$(284 + 48\sqrt{35}) - 24(12 + 2\sqrt{35}) + 4 = 0.$$

This implies that α satisfies $x^4 - 24x^2 + 4 = 0$.